STRINGQFT

String-Theoretic Construction
of Quantum Field Theory

VOLUME I

Supersymmetric Field Theory (2D-10D)

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Contents

I.1 Supersymmetry algebras and the dimensional ladder 1
I.1.1 Counting supersymmetry from the algebra . . . 1
I.1.2 Spinors and supercharge counting . . . 3
I.1.3 The dimensional ladder . . . 5
I.1.4 Central charges and BPS shortening . . . 9
I.1.5 Superconformal algebras and the unitarity bound . . . 10
I.1.6 Master tables and chapter boundary . . . 13
Status and sources . . . 17
Further reading . . . 19
References . . . 19

I.2 Common SUSY-QFT grammar: multiplets, local terms, vacua, anomalies 21
I.2.1 Multiplets are representations with local descendants . . . 22
I.2.2 F-terms, D-terms, and the first vacuum equations . . . 24
I.2.3 Branches, protected rings, and local coordinates . . . 27
I.2.4 Symmetries, currents, and anomaly traces . . . 29
I.2.5 Deformations and parameters . . . 31
I.2.6 The routing table for the dimension chapters . . . 33
I.2.7 Theory cards: the reading format . . . 35
I.2.8 Worked dictionary: one frame, many later uses . . . 37
Exit checklist . . . 38
Status and sources . . . 39
Further reading . . . 40
References . . . 40

I.3 $4d$ $\mathcal{N}=1$ supersymmetric field theories 42
I.3.1 Multiplets, holomorphy, and the dynamical scale . . . 43
I.3.2 The chiral ring and non-renormalization . . . 46
I.3.3 Pure super Yang–Mills and the index . . . 47
I.3.4 R-symmetry, NSVZ, and the superconformal R . . . 50
I.3.5 't Hooft anomalies and the Konishi anomaly . . . 51
I.3.6 a-maximization: the extremization prototype . . . 54
I.3.7 Seiberg duality and the conformal window . . . 60
I.3.8 Dynamical supersymmetry breaking and the index . . . 68

Exit checklist . . . 70
Status and sources . . . 71
Further reading . . . 73
References . . . 74

I.4 $2d$ supersymmetric field theories 76
I.4.1 Kinematics and R-symmetries . . . 77
I.4.2 The $\mathcal{N} = (2, 2)$ holomorphic sectors and their rings . . . 81
I.4.3 The GLSM, from CP^{N-1} to the quintic . . . 82
I.4.4 The A and B topological twists and mirror symmetry . . . 88
I.4.5 $\mathcal{N} = (0, 2)$ Fermi multiplets and the E/J data . . . 90
I.4.6 Bundles and $\mathcal{N} = (0, 2)$ deformations . . . 93
I.4.7 2d anomalies and c-extremization . . . 95
I.4.8 $\mathcal{N} = (0, 2)$ triality . . . 100
I.4.9 The elliptic genus . . . 104
I.4.10 Comparison of two-dimensional theories . . . 107
Exit checklist . . . 108
Status and sources . . . 109
Further reading . . . 113
References . . . 113

I.5 $4d$ $\mathcal{N} = 2$ supersymmetric field theories 115
I.5.1 $\mathcal{N} = 2$ multiplets in $\mathcal{N} = 1$ language . . . 116
I.5.2 Electromagnetic duality and the charge lattice . . . 117
I.5.3 The Coulomb branch and rigid special geometry . . . 119
I.5.4 Seiberg–Witten theory and the central charge . . . 121
I.5.5 Matter, decoupling, and BPS wall crossing . . . 127
I.5.6 The hyperkähler Higgs branch . . . 130
I.5.7 S-duality and the marginal coupling . . . 131
I.5.8 Conformal central charges of a Lagrangian theory . . . 132
I.5.9 Argyres–Douglas fixed points . . . 134
Exit checklist . . . 136
Status and sources . . . 137
Further reading . . . 140
References . . . 140

I.6 $3d$ supersymmetric field theories 143
I.6.1 $\mathcal{N} = 2$ multiplets and the dual photon . . . 144
I.6.2 The Coulomb branch and the topological $U(1)_J$ . . . 147
I.6.3 Monopole operators and their dimensions . . . 148
I.6.4 F-maximization . . . 151
I.6.5 Chern-Simons-matter theories and the parity anomaly . . . 154
I.6.6 ABJM and the $N^{3/2}$ free energy . . . 155
I.6.7 The Aharony and Giveon–Kutasov dualities . . . 157
I.6.8 The $\mathcal{N} = 4$ enhancement . . . 160
I.6.9 Three-dimensional mirror symmetry . . . 163

Exit checklist . . . 166
Status and sources . . . 167
Further reading . . . 170
References . . . 170

I.7 $5d$ supersymmetric field theories 174
I.7.1 Multiplets and the irrelevant coupling . . . 175
I.7.2 The real Coulomb branch . . . 177
I.7.3 The cubic prepotential and its metric . . . 178
I.7.4 Instanton operators and the E_n flavor enhancement . . . 184
I.7.5 The $5d$ superconformal fixed points . . . 187
I.7.6 The $5d$ superconformal index . . . 189
Exit checklist . . . 191
Status and sources . . . 191
Further reading . . . 193

I.8 $6d$ supersymmetric field theories 195
I.8.1 Multiplets and the self-dual tensor . . . 196
I.8.2 The tensor branch . . . 198
I.8.3 The anomaly polynomial . . . 200
I.8.4 Green–Schwarz–Sagnotti–West factorization . . . 204
I.8.5 Conformal anomalies on the tensor branch . . . 206
I.8.6 The maximal $\mathcal{N} = (2, 0)$ theory . . . 208
I.8.7 The $\mathcal{N} = (1, 1)$ maximal super-Yang–Mills . . . 211
Exit checklist . . . 212
Status and sources . . . 214
Further reading . . . 220
References . . . 220

I.9 $7d$ to $10d$ supersymmetric field theories 223
I.9.1 $10d$ $\mathcal{N} = 1$ super Yang–Mills: the parent . . . 224
I.9.2 The reduction sequence $10d \to 7d$ . . . 226
I.9.3 R-symmetry across dimensions $SO(10 - d)$ . . . 233
I.9.4 The Coulomb branch on reduction . . . 235
I.9.5 The per-dimension catalog and string or M-theory setting . . . 238
I.9.6 No superconformal field theory above six dimensions . . . 239
Exit checklist . . . 240
Status and sources . . . 241
Further reading . . . 243
References . . . 243

Chapter I.1

Supersymmetry algebras and the dimensional ladder

What to carry forward. A label such as $\mathcal{N} = 2$ acquires its physical meaning only after the spacetime dimension is specified. This chapter turns the super-Poincaré algebra into a way to count real supercharges, follow them through ordinary dimensional reduction, and recognize the R-symmetry and shortening conditions they allow. The master tables in §I.1.6 collect this dictionary for the theories used later in the book.

What this chapter delivers

The real supercharge count and spinor reality type in every dimension $d = 2, \ldots, 11$ (§I.1.2); the R-symmetry that rotates them (§I.1.2); the dimensional ladder that carries one count across many $(d, \mathcal{N})$ labels (§I.1.3); the BPS bound and the shortening that protects BPS data (§I.1.4); the superconformal ceiling that bounds where interacting fixed points live (§I.1.5); and the master tables that collect it all (§I.1.6).

I.1.1 Counting supersymmetry from the algebra

A $4d$ $\mathcal{N} = 2$ theory has eight real supercharges, while a $3d$ $\mathcal{N} = 2$ theory has four. The Lorentz spinor changes with dimension, so the number of copies named by $\mathcal{N}$ cannot by itself tell us the number of generators. We therefore begin with real counts: $2d$ $(0, 1)$ has one, $2d$ $(0, 2)$ and $3d$ $\mathcal{N} = 1$ have two, $4d$ $\mathcal{N} = 1$ has four, $4d$ $\mathcal{N} = 2$ and $5d$ $\mathcal{N} = 1$ have eight, and $4d$ $\mathcal{N} = 4$ has sixteen. The notation changes with dimension, while an ordinary supersymmetry-preserving reduction keeps the count.

Volume 0 develops the supersymmetric field-theory structures used by later constructions. Here the algebra determines the count and its dimensional translations. Subsequent chapters supply the multiplets, dynamics, and examples associated with each dimension.

Two things this volume is not

(i) Volume 0 is not a comprehensive quantum field theory textbook. It teaches the supersymmetry toolkit the rest of the book *uses*, routing the full proofs and dynamics to their later homes. **(ii)** The label $\mathcal{N}$ (or (p,q)) does not denote the same amount of supersymmetry across dimensions: the invariant along the dimensional ladder is the real supercharge *count*, not the label (§I.1.3).

The super-Poincaré anticommutator does more than name a symmetry. Its spinor indices determine the possible real supercharges; its positive expectation values bound the mass of charged states. After counting the spinors, we follow the same generators into lower dimensions and then use positivity again for BPS states and conformal primaries.

The super-Poincaré algebra.

Under the usual assumptions on a nontrivial scattering theory, the graded extension of Poincaré symmetry is tightly constrained. The Coleman–Mandula theorem forbids the Poincaré generators from combining nontrivially with any internal symmetry through *bosonic* charges. The supercharges evade it because they are *fermionic*: they transform as spinors of the Lorentz group. The Haag–Lopuszański–Sohnius theorem then shows that the resulting graded extension is essentially unique. We work throughout in Lorentzian signature $\eta = \mathrm{diag}(-,+,\ldots,+)$ with the Lorentzian spinor conventions stated here.

A supercharge Q is a spinor of $\mathrm{Spin}(1,d-1)$. The defining relation of the algebra is the anticommutator of two supercharges, which closes on the translations,

$$\{Q_\alpha,\, \bar{Q}_{\dot\beta}\} = 2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu, \tag{I.1.1}$$

written here in the four-dimensional two-component form for definiteness; in general d the right-hand side is $(C\Gamma^\mu)_{\alpha\beta}P_\mu$ with C the charge-conjugation matrix. The supercharges commute with the momentum, $[P_\mu, Q_\alpha] = 0$, and rotate as spinors under the Lorentz generators, $[M_{\mu\nu}, Q_\alpha] = -(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$. In a unitary representation, the expectation value of an anticommutator of a supercharge with its adjoint is nonnegative. The matrix $2\sigma^\mu P_\mu$ in (I.1.1) inherits that positivity in the appropriate state and frame; it leads to the energy bound in §I.1.4. The spinor indices on the left require reality and chirality conditions, which fix the count in §I.1.2.

Extended supersymmetry and central charges. With $\mathcal{N}$ supercharges Q^I, $I = 1,\ldots,\mathcal{N}$, the algebra admits an extension that the minimal case does not:

$$\{Q^I_\alpha,\, \bar{Q}_{\dot\beta\, J}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,\delta^I_J, \qquad \{Q^I_\alpha,\, Q^J_\beta\} = \epsilon_{\alpha\beta}\,Z^{IJ}, \tag{I.1.2}$$

where $Z^{IJ} = -Z^{JI}$ is a Lorentz-scalar *central charge*, commuting with the super-Poincaré generators. Higher-dimensional algebras can also carry antisymmetric tensorial charges $Z_{\mu_1\cdots\mu_p}$. These transform under Lorentz rotations and are not central in the strict sense; they measure charges of extended objects. For a particle with scalar central charge, positivity relates its mass to the charge in §I.1.4. In four-dimensional Seiberg–Witten theory that charge combines electric and magnetic quantum numbers with Coulomb-branch coordinates.

The Nahm ceiling. With a bound on the spin content of particle multiplets, representation theory also restricts the number of supercharges. A multiplet of a super-Poincaré algebra with more than 32 real supercharges necessarily contains fields of spin greater than two, while 32 real supercharges already force a spin-two field in a massless multiplet. This is the physical ceiling used in field theory: $d = 11$ is the top gravity-coupled case with a 32-component minimal spinor, and a formal $d = 12$ spinor would carry 64 real components and force higher-spin states. We state this Nahm ceiling and use it, citing the proof rather than reproducing it (§I.1.6). It is the reason the dimensional ladder treats $32Q$ as an ambient ceiling rather than an ordinary row of decoupled QFTs.

I.1.2 Spinor reality, R-symmetry, and supercharge accounting

How many real supercharges does the minimal spinor carry in dimension d? The answer is fixed by the reality structure of a spinor of $\mathrm{Spin}(1, d-1)$. That structure comes from the real Clifford algebra, and it repeats with period eight in d (the Bott periodicity of the real Clifford algebras). A spinor can carry up to two independent reductions of its components:

- a *Majorana* (M) condition, a reality condition $\psi^* = B\psi$ compatible with the Lorentz action, available when the charge-conjugation matrix squares the right way; it halves the real dimension;
- a *Weyl* (W) condition, a chirality projection $\Gamma_*\psi = \pm\psi$, available in even d; it also halves the dimension.

When both conditions hold at once the spinor is Majorana–Weyl (MW), and it is quartered. When the natural reality condition pairs two spinors instead of constraining one, the spinor is symplectic-Majorana (SM); if a chirality also survives, it is symplectic-Majorana–Weyl (SMW). Which of {M, W, MW, SM, SMW} occurs in dimension d is fixed by $(d-2)$ mod 8. The minimal real supercharge count then follows from the type and the complex Dirac dimension $2^{\lfloor d/2 \rfloor}$:

$$\begin{aligned} &\text{M: } 2^{\lfloor d/2 \rfloor}; \quad \text{MW: } 2^{\lfloor d/2 \rfloor - 1} \text{ (per chirality);} \\ &\text{SM: } 2^{\lfloor d/2 \rfloor + 1}; \quad \text{SMW: } 2^{\lfloor d/2 \rfloor} \text{ (per chirality).} \end{aligned} \tag{I.1.3}$$

Evaluating the type and (I.1.3) dimension by dimension gives the minimal real supercharge count, the first column every later chapter reads:

d	2	3	4	5	6	7	8	9	10	11
type	MW	M	M	SM	SMW	SM	M	M	MW	M
min Q	1	2	4	8	8	16	16	16	16	32

(I.1.4)

The pattern is not a list to memorize. The allowed reality condition changes with the real Clifford algebra, while the Dirac spinor grows with dimension. The count is *not* strictly increasing in d: it is 16 for $d = 7, 8, 9, 10$ and jumps to 32 only at $d = 11$,

the gravity-coupled ceiling. The $d = 8$ case also has a 16-real-component minimal spinor but no Majorana–Weyl spinor. Equivalently, the Weyl spinor there is complex, and the real Majorana package is not chiral. So the naive expectation "even d has Majorana–Weyl spinors" is wrong: the reality type decides, not the parity of d. The count records the same content either way. In four dimensions the four real components are often packaged as a complex Weyl supercharge plus its conjugate; the table labels the equivalent real Majorana package.

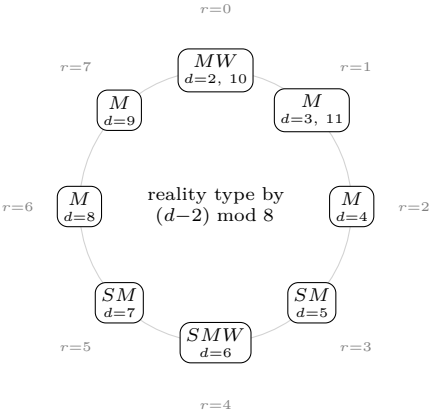


Figure I.1.1: The spinor reality type is periodic in $(d-2)$ mod 8 (Bott periodicity of the real Clifford algebra), so it is fixed by the residue r alone. Dimensions $d = 2, \ldots, 9$ represent the eight residue classes; $d = 10$ has the reality type of $d = 2$, and $d = 11$ that of $d = 3$. The minimal real *count*, the type applied to the Dirac dimension $2^{\lfloor d/2 \rfloor}$, is *not* periodic: it grows with d even where the type repeats.

An amount of supersymmetry with $\mathcal{N}$ copies of the minimal spinor, or with chirality content (p, q) in the even dimensions that admit it, carries $\mathcal{N}$ (respectively $p + q$) times the minimal count of (I.1.4). That product is the single number the next two sections organize.

R-symmetry.

The $\mathcal{N}$ supercharges of an extended algebra can be rotated into one another by a global symmetry that commutes with the Lorentz group and acts on the index I of (I.1.2). This is the *R-symmetry*: the automorphism group of the supersymmetry algebra,

$$[\, R_a,\, Q^I_\alpha \,] = (t_a)^I{}_J\, Q^J_\alpha, \tag{I.1.5}$$

with t_a the generators in the representation carried by the supercharges. As a first mnemonic, real, complex, and symplectic supercharge packages are rotated respectively by orthogonal, unitary, and compact symplectic automorphism groups. The actual R-symmetry of a physical or superconformal theory is dimension-dependent, and can be reduced by chirality, central charges, or projectivization. The master R-symmetry table below (Table I.1.5) is the authoritative lookup used by the rest of the book. At a superconformal fixed point, a particular R-current enters the larger algebra of

§I.1.5. When the ultraviolet theory has several abelian currents, identifying that current requires the appropriate extremization principle: a-maximization in $4d$, F-maximization in $3d$, or c-extremization in $2d$.

The extremes are as important as the middle of the ladder. At the very bottom, a purely two-dimensional chiral algebra can have just one real supercharge: $2d$ $(0,1)$ or $(1,0)$ has a single Majorana–Weyl generator and no continuous R-symmetry, only the discrete sign automorphism of that generator. With two real supercharges the R-symmetry can still be small or absent. The $3d$ $\mathcal{N}=1$ and $2d$ $(1,1)$ algebras have no continuous R-symmetry that acts as a superconformal R-current. The chiral $2d$ $(0,2)$ algebra does: it has a right-moving $U(1)_R$. With sixteen real supercharges the R-symmetry is large enough to dominate the representation theory. Here $4d$ $\mathcal{N}=4$ has $SU(4)_R \simeq \mathrm{Spin}(6)$, and $3d$ $\mathcal{N}=8$ superconformal theories have $\mathrm{Spin}(8)_R$ (the maximal-SYM Lagrangian shows only the scalar-rotation Spin(7), enhanced in the infrared). The high-dimensional rows are smaller. The $8d$ $\mathcal{N}=1$ row has $U(1)_R$, from the two transverse directions to a 7-brane, and $7d$ $\mathcal{N}=1$ has $SU(2)_R$. The $5d$ maximal super Yang–Mills and $6d$ $(2,0)$ rows have $USp(4)_R \simeq \mathrm{Spin}(5)_R$, and $6d$ $(1,1)$ has the $\mathrm{Spin}(4)_R$ of its nonchiral sixteen supercharges. These high-symmetry rows are not decorative. They are the reason the maximally supersymmetric constructions are so rigid. They are also the rows from which maximal-SYM and sixteen-supercharge fixed-point discussions read their protected data.

Three identifications recur and are worth fixing against their common confusions. In four dimensions, $\mathcal{N}=2$ has R-symmetry $SU(2)_R \times U(1)_R$, not $SU(3)$: the two Weyl supercharges Q^I_α, carrying eight real components in total, form an $SU(2)_R$ doublet and carry a $U(1)_R$ charge. That automorphism algebra has dimension four and rank two; $SU(3)$ has dimension eight, too large to act on a doublet. In two dimensions, the $(0,2)$ algebra has a single $U(1)_R$, the right-moving R-symmetry. The left-moving $U(1)_L$ that often accompanies it is a *flavor* current, used in c-extremization, not an R-symmetry factor. In five dimensions, $\mathcal{N}=1$ has R-symmetry $SU(2)_R$. The $U(1)_I$ instanton symmetry that enhances to E_n at the ultraviolet fixed point is a *flavor* symmetry, not an R-symmetry. Each of these is a row of the R-symmetry table (Table I.1.5). Their Lie-algebra ranks and dimensions distinguish the proposed groups. The remaining cells follow the same rule: $3d$ $\mathcal{N}=4$ has $SU(2) \times SU(2)$ (the algebra $\mathfrak{so}(4)$), and $6d$ $(2,0)$ has $USp(4)$, the spin cover Spin(5) of $SO(5)$, of dimension ten and rank two.

I.1.3 The dimensional ladder

For an ordinary circle reduction with supersymmetry-preserving boundary conditions and no projection, the higher-dimensional supercharges yield the same number of lower-dimensional generators. Their Lorentz representation changes, so the *label* $\mathcal{N}$ changes while the real count stays fixed. A reduction with a twist or projection need not preserve them all. Even when two theories share a count, their chirality and R-symmetry can differ. The following table therefore separates reduction chains from same-count chiral variants.

Table I.1.1 displays the named rungs the book uses most often, not every possible label in every dimension. The exceptional first row is possible because a $2d$ Majorana–

real Q	balanced reduction ladder	chiral / same-count variants
1	purely $2d$	$2d$ $(0,1)$ or $(1,0)$; the minimal chiral field-theory row
2	$3d$ $\mathcal{N}{=}1 \to 2d$ $(1,1)$	$2d$ $(0,2)$ or $(2,0)$; the minimal chiral gauge-theory row
4	$4d$ $\mathcal{N}{=}1 \to 3d$ $\mathcal{N}{=}2 \to 2d$ $(2,2)$	$2d$ $(0,4)$ is same count, different chirality
8	$6d$ $(1,0) \to 5d$ $\mathcal{N}{=}1 \to 4d$ $\mathcal{N}{=}2 \to 3d$ $\mathcal{N}{=}4 \to 2d$ $(4,4)$	$2d$ $(0,8)$ chiral cousins; hypermultiplet row
16	$10d$ $\mathcal{N}{=}1$ SYM $\to 9d$ $\mathcal{N}{=}1 \to 8d$ $\mathcal{N}{=}1 \to 7d$ $\mathcal{N}{=}1$ $\to 6d$ $(1,1) \to 5d$ $\mathcal{N}{=}2 \to 4d$ $\mathcal{N}{=}4 \to 3d$ $\mathcal{N}{=}8 \to 2d$ $(8,8)$	$6d$ $(2,0)$ is chiral 16Q, not an ordinary Yang–Mills Lagrangian
32	$11d$ super-Poincaré / ten-dimensional type-II algebra	ceiling for gravity-coupled algebras, not a Volume 0 decoupled-QFT row

Table I.1.1: The dimensional ladder by real supercharge count. The one-supercharge row is genuinely two-dimensional; later arrows are ordinary circle reductions, so the count is conserved while the dimension-dependent label changes. Same-count rows with different chirality are not interchangeable.

Weyl spinor has one real component per chirality: a $(0,1)$ or $(1,0)$ theory is a one-supercharge theory, not a shorthand for a hidden two-supercharge algebra. The complete low-dimensional field-theory list has the same simple arithmetic behind it. In three dimensions $\mathcal{N} = 1, \ldots, 8$ carry $2, 4, \ldots, 16$ real supercharges. In four dimensions $\mathcal{N} = 1, 2, 3, 4$ carry $4, 8, 12, 16$. In five dimensions $\mathcal{N} = 1, 2$ carry $8, 16$. In six dimensions the standard rows are $(1,0)$, $(1,1)$, and $(2,0)$. Seven through ten dimensions each have the $\mathcal{N} = 1$ sixteen-supercharge pure-SYM row. Two dimensions are more flexible, because the left- and right-moving supercharges can be chosen independently: (p, q) has $p + q$ real supercharges. The count table below records the compact version of this list; the count itself always comes from (I.1.4).

The one-supercharge row is the first place where the count is too small to hide behind any higher-dimensional notation. It is a genuinely two-dimensional possibility: a $2d$ $(0,1)$ theory has one right-moving supercharge, and the left-moving $(1,0)$ version is its parity mirror. The two-supercharge row is the first row connected to ordinary higher-dimensional reduction. In three dimensions $\mathcal{N} = 1$ means two real supercharges; reducing it on a circle gives $2d$ $(1,1)$, again two real supercharges. The chiral $2d$ $(0,2)$ algebra also has two real supercharges, but it is not just the reduction of the balanced $(1,1)$ theory. Its right-moving chirality, its Fermi multiplets, and its $U(1)_R$ are genuine two-dimensional field-theory data, which is why a $(0,2)$ gauge theory is not a disguised $(2,2)$ theory. The lesson is already visible here: a supercharge count is necessary information, but it is not a complete theory label.

The four-supercharge ladder collects

$$4d\ \mathcal{N}{=}1 \ \cong\ 3d\ \mathcal{N}{=}2 \ \cong\ 2d\ (2,2) \ =\ 4 \text{ real supercharges}, \tag{I.1.6}$$

This is the standard four-supercharge field-theory row: four supercharges are just enough for chiral multiplets, holomorphic superpotentials, chiral rings, and a-maximization in

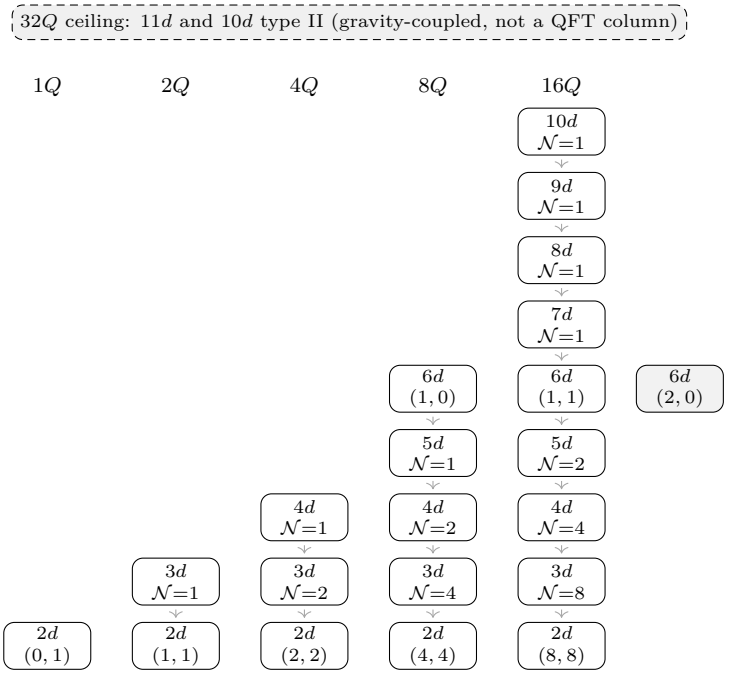


Figure I.1.2: The dimensional ladder by real supercharge count. Each column is one count; a circle reduction moves *down* a column (arrows), conserving the count while the $(d, \mathcal{N})$ label changes. Chiral cousins carry the same count by a different chirality: the $6d\,(2,0)$ theory beside the non-chiral $6d\,(1,1)$, and the $2d\,(p,q)$ rows. The $32Q$ row is the gravity-coupled ceiling, not a decoupled-QFT column. This is the picture behind Table I.1.1.

four dimensions, but not enough for the rigidity of the eight- or sixteen-supercharge rows.

The eight-supercharge ladder collects

$$6d\ (1,0)\ \cong\ 5d\ \mathcal{N}{=}1\ \cong\ 4d\ \mathcal{N}{=}2\ \cong\ 3d\ \mathcal{N}{=}4\ \cong\ 2d\ (4,4)\ =\ 8 \text{ real supercharges}, \tag{I.1.7}$$

This is the hypermultiplet row: hypermultiplets, Coulomb-Higgs factorization, four-dimensional Coulomb-branch low-energy data, and enough protection that many statements about the low-energy effective theory become exact.

The sixteen-supercharge row is a different animal. One branch is the maximally supersymmetric Yang–Mills ladder,

$$\begin{aligned} 10d\ \mathcal{N}{=}1 \text{ SYM} \to 9d\ \mathcal{N}{=}1 \to 8d\ \mathcal{N}{=}1 \to 7d\ \mathcal{N}{=}1 \to 6d\ (1,1) \\ \to 5d\ \mathcal{N}{=}2 \to 4d\ \mathcal{N}{=}4 \to 3d\ \mathcal{N}{=}8 \to 2d\ (8,8), \end{aligned} \tag{I.1.8}$$

all with 16 real supercharges. Another branch with the same count is the chiral $6d\ (2,0)$ theory: it has no ordinary Yang–Mills Lagrangian in six dimensions, yet it carries

sixteen real supercharges and sits on the same count ladder, its circle reduction landing on the five-dimensional maximal theory. It is not "more of the same" as the eight-supercharge row; the larger R-symmetry and the absence or presence of a Lagrangian are load-bearing field-theory data.

The high-dimensional end of this sequence is still field theory. Ten-dimensional $\mathcal{N} = 1$ SYM is the parent pure gauge theory; reducing it gives legitimate non-gravitational $9d$, $8d$, and $7d$ $\mathcal{N} = 1$ SYM theories. None of these sixteen-supercharge rows is forced to contain a graviton. The $8d$ row describes the gauge theory on a 7-brane in F-theory constructions; the neighboring $7d$ $\mathcal{N} = 1$ row enters constructions with G_2 geometry. Their sixteen-supercharge counts follow from the same ten-dimensional parent.

There is one useful field-theory check on this row, and it is a short derivation rather than a slogan. A propagating minimal pure super Yang–Mills Lagrangian has only a gauge field and one gaugino, so it has the familiar on-shell degree-of-freedom match only where their physical polarizations agree. A massless vector in d dimensions has $d - 2$ physical polarizations, while the minimal gaugino carries half the minimal real supercharge count of (I.1.4) on shell (the Dirac equation halves the off-shell components); equating the two,

$$d - 2 \;=\; \tfrac{1}{2}\, Q_{\min}(d) \quad \Longrightarrow \quad d - 2 \in \{1, 2, 4, 8\}, \qquad d \in \{3, 4, 6, 10\}, \tag{I.1.9}$$

since $Q_{\min} = 2, 4, 8, 16$ at $d = 3, 4, 6, 10$ gives $\frac{1}{2}Q_{\min} = 1, 2, 4, 8 = d - 2$, and no other dimension matches. These four are the *minimal* super-Yang–Mills theories; reducing the ten-dimensional one on a torus generates the entire maximal-SYM ladder (I.1.8). The reduction also fixes each rung's R-symmetry. Reducing $10d$ $\mathcal{N} = 1$ SYM to D dimensions leaves $10 - D$ adjoint scalars. The $SO(10 - D)$ that rotates them is the R-symmetry of the maximal theory. It acts directly on the scalars; on the gaugino, and hence on the supercharges, it acts through its spin cover. The group acting on the *supercharges* is therefore $\mathrm{Spin}(10-D)$. For $D = 8$ this is $SO(2) \simeq U(1)_R$. For $D = 7$ the scalars rotate by $SO(3)$, and the supercharges feel its double cover $SU(2)_R = \mathrm{Spin}(3)$. This is the $SU(2)_R$ of the master table, *not* a quotient of it. The pattern continues down to $D = 3$, where the Lagrangian shows only $\mathrm{Spin}(7)$. If the infrared limit exists and is superconformal, its R-symmetry is $\mathrm{Spin}(8)$.

Finally, the thirty-two-supercharge ceiling is the algebraic upper edge. It is the spinor size of eleven-dimensional super-Poincaré symmetry and of the type-II ten-dimensional algebras, and it tells us why the ladder stops. It is not a row of ordinary decoupled QFTs inside Volume 0. With that many supercharges one is at the gravity-coupled ceiling, not at a stand-alone supersymmetric field theory in the sense this volume teaches. Going to $64Q$, as in a formal twelve-dimensional spinor algebra, would force higher-spin states.

The invariant under the ordinary reduction is the count, not the symbol $\mathcal{N}$. Four examples make the point. Four-dimensional $\mathcal{N} = 2$ and three-dimensional $\mathcal{N} = 2$ are *different* amounts, eight against four, because the minimal spinor halves between the two dimensions. Five-dimensional $\mathcal{N} = 1$ and four-dimensional $\mathcal{N} = 2$ are the *same* amount. Two-dimensional $(0, 1)$ has no three-dimensional parent with its single count. Two-dimensional $(0, 2)$ and three-dimensional $\mathcal{N} = 1$ share a count but not a chirality. To compare two theories, state the real count first and then specify the dimension, chirality, and R-symmetry.

Worked example: embeddings preserve Q

A transverse orbifold embedding can select a row of the count table before one constructs a quiver. Start with the D3-brane parent, which has sixteen real supercharges. The transverse orbifold group Γ decides how many survive. The surviving supercharges are those left invariant by Γ acting on the four-component transverse spinor (the $\mathbf{4}$ of the transverse $\mathrm{Spin}(6) \simeq SU(4)$), and each invariant component carries four real supercharges. If $\Gamma \subset SU(2)$, one complex transverse direction is inert and two of those components stay invariant ($\mathbf{4} \to \mathbf{2} + \mathbf{1} + \mathbf{1}$): the four-dimensional probe theory keeps eight of the sixteen supercharges, namely $4d\ \mathcal{N} = 2$. This is the $\mathbb{C}^2/\Gamma$ ADE block. If Γ lies generically in $SU(3)$, only one component stays invariant ($\mathbf{4} \to \mathbf{3} + \mathbf{1}$): the probe theory keeps four real supercharges, namely $4d\ \mathcal{N} = 1$. This is the $\mathbb{C}^3/\Gamma$ chiral-quiver block. If an $SU(3)$ embedding accidentally factors through an $SU(2)$ subgroup, the count jumps back to eight real supercharges. The embedding thus chooses the row of the ladder before any quiver or superpotential data is read. The routing is by complex dimension, not by one volume: $\mathbb{C}^2/\Gamma$ orbifold probes are Volume V chs. 3–5 ($8Q$, $4d\ \mathcal{N} = 2$); $\mathbb{C}^3/\Gamma$ orbifold probes are Volume V ch. 6 ($4Q$, $4d\ \mathcal{N} = 1$); the general toric CY3 and dimer machinery is Volume VI.

I.1.4 Central charges, the BPS bound, and shortening

For a charged massive state, positivity of the supersymmetry anticommutator constrains its rest mass. Saturation produces a shorter representation and an exact relation between mass and charge. In a single central-charge sector, choose a normalization of the charge so that the anticommutator acting on a one-particle state of mass M has the Hermitian block

$$A = \begin{pmatrix} M & Z \\ \bar{Z} & M \end{pmatrix}, \qquad \text{eigenvalues} \quad M \pm |Z|. \tag{I.1.10}$$

Because $A = \{Q, Q^\dagger\} \geq 0$ is a sum of squares, both eigenvalues are nonnegative, which is the *BPS bound*

$$M \ \geq \ |Z|. \tag{I.1.11}$$

A massive state saturating the bound, $M = |Z| > 0$, makes $\det A = M^2 - |Z|^2$ vanish: the block drops rank, and one of the two combinations in this block annihilates the state. These *preserved* supercharges are the ones for which $\{Q, Q^\dagger\} = M - |Z| = 0$; the remaining *broken* supercharges, with eigenvalue $2M$, act nontrivially and build a multiplet that is *shortened*, with fewer states. To count them, consider a massive $4d\ \mathcal{N} = 2$ representation at fixed charge and with a spin-zero Clifford vacuum. Its two spin-polarization blocks provide four complex creation operators above the bound and two at saturation. Each independent fermionic creation operator can act either zero or one time, so r such operators generate 2^r states. The long and short representations therefore have $2^4 = 16$ and $2^2 = 4$ states, respectively. Halving the active supercharges has reduced the state count by a factor of four in this example. Adding a distinct CPT-conjugate charge sector doubles both counts; a different Clifford-vacuum spin changes the comparison and must be specified. As long as a state remains in that short

representation, its mass follows the algebraic charge relation. The algebra alone does not ensure that the same stable one-particle state exists throughout moduli space. At a wall of marginal stability it can decay, while short multiplets can recombine when the missing states become available. An index or a BPS degeneracy must therefore be interpreted in its chamber; the susceptibility to recombination depends on the fraction of preserved supersymmetry. This distinction lets us use exact mass relations without assuming a fixed spectrum over every deformation.

The canonical instance comes from the Seiberg–Witten solution of four-dimensional $\mathcal{N} = 2$ super Yang–Mills, in the normalization used throughout these notes. There the central charge is the holomorphic combination of the electric and magnetic charges (n_e, n_m),

$$Z \;=\; a\, n_e \;+\; a_D\, n_m, \qquad M_{\mathrm{BPS}} \;=\; \sqrt{2}\,|Z|, \tag{I.1.12}$$

with a and a_D the special coordinate and its dual on the Coulomb branch. The Z in (I.1.12) follows the book's Seiberg–Witten convention. The schematic block (I.1.10) instead uses a charge normalized so its positivity bound is $M \geq |Z|$; for this example its off-diagonal entry is $\sqrt{2}Z$ as defined in (I.1.12). For a purely electric BPS state, $n_m = 0$, the relation gives $M = \sqrt{2}\,|a|\,|n_e|$ if that state is present; a state with $n_m \neq 0$ carries magnetic charge. Chapter 1 states the bound, the shortening mechanism, and this canonical central charge. It computes no specific spectrum. Which states are present, and how a and a_D vary, is the dynamics supplied by Volume V's $\mathcal{N} = 2$ constructions and Part III. The block (I.1.10) shows why saturation removes one combination of supercharges; the dynamics must determine which charges and states occur.

I.1.5 Superconformal algebras and the unitarity bound

Some of the theories constructed later flow to conformal fixed points. There the Poincaré symmetry enhances to the conformal group $SO(d, 2)$, and the supersymmetry algebra enlarges with it. This section does for the fixed point what §I.1.4 did for the massive spectrum. The same positivity that turned $\{Q, Q^\dagger\} \geq 0$ into a mass bound now bounds scaling dimensions. The operators that saturate the bound are the protected, short spectrum, and a superconformal theory computes them exactly. This algebraic relation underlies the fixed-point quantities computed in later chapters.

From super-Poincaré to superconformal. The conformal group adds to the Poincaré generators $(P_\mu, M_{\mu\nu})$ the dilatation D and the special conformal generators K_μ. Supersymmetrizing it cannot stop at the Poincaré supercharges Q: closing Q with K forces a second fermionic generator, the *superconformal supercharge* S, generated by the commutator $[K_\mu, Q]$. It is the dimension-lowering partner of Q,

$$[\,D,\, Q\,] \;=\; +\tfrac{1}{2}\, Q, \qquad\qquad [\,D,\, S\,] \;=\; -\tfrac{1}{2}\, S, \tag{I.1.13}$$

so Q raises the scaling dimension by half a unit and S lowers it. The three fermionic brackets then close on the three bosonic sectors; in the standard conventions they are

$$\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu, \qquad \{S, \bar{S}\} = 2\sigma^\mu K_\mu, \qquad \{Q_\alpha, S^\beta\} = (\sigma^{\mu\nu})_\alpha{}^\beta\, M_{\mu\nu} + \delta_\alpha^\beta\,(D - c\,R), \tag{I.1.14}$$

The first is the super-Poincaré relation (I.1.1) itself, and the second is its conformal conjugate ($P \to K$). The third is the decisive one. Its Lorentz part $(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}$ annihilates a scalar primary. What remains on the diagonal is the combination $D - cR$ of the dilatation and the R-symmetry, with c the algebra constant ($c = \frac{3}{2}$ for $4d\ \mathcal{N} = 1$). At a fixed point the R-symmetry is therefore no longer an external automorphism of the algebra (§I.1.2) but a *generator inside it*, graded next to the dilatation. That single fact makes the R-symmetry load-bearing at a fixed point, and it fixes the dimension of a short operator once its R-charge is known.

The unitarity bound and BPS shortening at the fixed point. Take a superconformal primary $|\mathcal{O}\rangle$, an operator annihilated by every S (the conformal analog of a lowest-weight state), with scaling dimension Δ and R-charge R. Exactly as in §I.1.4, the relevant supercharge pairing is a positive Hermitian operator. In radial quantization the conformal conjugate of Q is precisely $S = Q^\dagger$, and a superconformal primary is annihilated by every S, so the diagonal matrix element collapses to a single norm,

$$\langle\mathcal{O}|\,\{Q,\,S\}\,|\mathcal{O}\rangle \;=\; \langle\mathcal{O}|\,Q^\dagger Q\,|\mathcal{O}\rangle \;=\; \big\|\,Q|\mathcal{O}\rangle\,\big\|^2 \;\geq\; 0, \tag{I.1.15}$$

where the $QS = QQ^\dagger$ term dropped because $S|\mathcal{O}\rangle = 0$ on a primary. By (I.1.14), the left-hand side equals the eigenvalue $\Delta - c\,R$ (the Lorentz part vanishes on a scalar). Reading it off gives a *superconformal shortening bound*, the fixed-point counterpart of $M \geq |Z|$,

$$\Delta \;\geq\; c\,R \qquad \text{(scalar primary, the chosen } \{Q,S\} \text{ branch)}, \tag{I.1.16}$$

with c the same algebra constant as in (I.1.14). This is one BPS branch. The full unitarity bounds for a generic (long) primary carry extra spin-dependent offsets; those are the representation-theoretic statement classified in the cited literature. A primary that *saturates* the bound, $\Delta = cR$, is annihilated by the corresponding supercharge. It sits in a *short* (BPS) representation, the exact analog of the rank drop of (I.1.10). In $4d\ \mathcal{N} = 1$ the saturating primaries are the *chiral* operators, annihilated by all $\bar{Q}$; the conjugate positivity $\{\bar{Q}, \bar{Q}^\dagger\} \geq 0$ then locks their dimension to the R-charge,

$$\Delta \;=\; \tfrac{3}{2}\,R \qquad (4d\ \mathcal{N} = 1 \text{ chiral primary}), \tag{I.1.17}$$

in the normalization where a chiral superpotential carries $R = 2$ (so that $R > 0$ on a chiral primary). This is the fixed-point version of $M = |Z|$. The dimension of a chiral/BPS operator is fixed by an algebraic charge, and it stays fixed under any deformation that preserves the short representation. That is what lets a superconformal theory pin anomalous dimensions perturbation theory cannot reach. The same graded caution as in §I.1.4 applies: the protection is the representation-theoretic shortening, and weaker short multiplets can recombine.

A worked instance and the extremization principle. The smallest check of (I.1.17) fixes the R-normalization from a marginal cubic. Requiring a cubic superpotential $W = \Phi^3$ to carry R-charge two (so that $\int d^2\theta\, W$ is R-neutral) forces $R(\Phi) = \frac{2}{3}$, hence

$$\Delta(\Phi) \;=\; \tfrac{3}{2}\,R(\Phi) \;=\; \tfrac{3}{2}\cdot\tfrac{2}{3} \;=\; 1, \tag{I.1.18}$$

the free-field dimension, consistent with the cubic having classical dimension three ($\Delta(W) = 3$, marginal at the free point). A free chiral primary thus saturates the bound

(I.1.16), while a non-chiral operator (a real scalar of $R = 0$ and $\Delta > 0$) does not. This checks the normalization of the $4d$ $\mathcal{N} = 1$ chiral-primary relation. It is not a claim that the single-field cubic Wess–Zumino model defines a nontrivial four-dimensional interacting fixed point. In an interacting theory the superconformal R is not put in by hand. It is a specific combination of all the abelian symmetries, *selected* by an extremization principle: a-maximization in $4d$, F-maximization in $3d$, c-extremization in $2d$. The particular extremization calculation depends on the theory and appears with the relevant constructions in Volume V and Part III (Table I.1.6). The whole derivation is one positivity argument used twice, at a massive state in §I.1.4 and at a fixed point here; Table I.1.2 reads the two off side by side.

	massive (super-Poincaré, §I.1.4)	fixed point (superconformal, §I.1.5)
positive operator	$\{Q, Q^\dagger\} = A \geq 0$	$\langle\mathcal{O}\vert\{Q, S\}\vert\mathcal{O}\rangle \geq 0$
bound	$M \geq \vert Z\vert$	$\Delta \geq cR$
saturation	$M = \vert Z\vert$ (half-BPS)	$\Delta = cR$ (chiral, short)
mechanism	preserved Q annihilate the state	$Q\vert\mathcal{O}\rangle = 0$, a short representation
protected quantity	BPS mass / index	dimension, $\Delta = \frac{3}{2}R$ at $4d$ $\mathcal{N}$=1

Table I.1.2: The BPS bound and the superconformal shortening bound are the same positivity argument, applied to a massive one-particle state (§I.1.4) and to a conformal primary (§I.1.5). The eigenvalues, the saturation, and the kernel-at-saturation match row by row.

The classification ceiling $d \leq 6$. Superconformal algebras are far more constrained than super-Poincaré ones. Requiring the fermionic generators to close into the conformal group as in (I.1.14), and the whole to admit unitary representations, the Nahm classification of superconformal algebras leaves only a finite list, and it terminates: *interacting* superconformal field theories exist only for $d \leq 6$, the four families and their boundary collected in Table I.1.3. The two-dimensional case is special: local superconformal symmetry there is governed by the super-Virasoro algebras, not by one of the finite-dimensional Nahm families below. Table I.1.3 records the finite-dimensional $d \geq 3$ superconformal algebras and the $d \geq 7$ ceiling; the $2d$ chapter develops super-Virasoro symmetry.

Table I.1.3 makes the consequences sharp. The $5d$ algebra has only eight supercharges, so the $5d$ $\mathcal{N} = 2$ sixteen-supercharge row is a maximal-SYM row, not a superconformal one. In $4d$ and $6d$ no superconformal theory has more than sixteen supercharges, and in $3d$ a theory with more than sixteen is free. Above six dimensions there is no interacting superconformal field theory at all. We state this classification and cite its proof (§I.1.6); the $d \leq 6$ ceiling is Nahm's. The argument has the same spirit as the positivity behind the super-Poincaré spin ceiling of §I.1.1. The gap between the two ceilings is the room

d	superconformal algebra	bosonic part (conformal $\times$ R)	max interacting Q	note
3	$\mathrm{OSp}(\mathcal{N}\|4)$	$Sp(4) \times SO(\mathcal{N})$	16 ($\mathcal{N}$=8)	$\mathfrak{so}(3,2)$=$\mathfrak{sp}(4)$; $\mathcal{N}$>8 free
4	$SU(2,2\|\mathcal{N})$	$SU(2,2) \times U(\mathcal{N})$	16 ($\mathcal{N}$=4)	$\mathfrak{su}(2,2)$=$\mathfrak{so}(4,2)$; $\mathcal{N}$=4 projectivized
5	$F(4)$	$SO(5,2) \times SU(2)$	8 ($\mathcal{N}$=1)	the unique $5d$ superconformal algebra
6	$\mathrm{OSp}(8^*\|2\mathcal{N})$	$SO(6,2) \times USp(2\mathcal{N})$	16 $\big((2,0)\big)$	chiral $(\mathcal{N},0)$; $(2,0)$ is $\mathcal{N}$=2
≥ 7	none	–	–	no interacting superconformal field theory

Table I.1.3: Interacting superconformal algebras by dimension (Nahm's classification). The conformal factor is $\mathfrak{so}(d,2)$ and the R-symmetry factor is the automorphism of §I.1.2, now graded inside the algebra. The maximal interacting supercharge count is read against the counts of §I.1.3; the algebra names and bosonic parts are the stated classification. There is no interacting superconformal field theory above six dimensions.

in which the gravity-coupled and the non-conformal supersymmetric theories live.

I.1.6 Master tables and chapter boundary

The chapter's deliverable is the navigation map in Tables I.1.4–I.1.6. The tables let the reader translate a supersymmetry label into a real count and then find the associated R-symmetry, multiplets, moduli branches, and protected quantities. The split is intentional: Table I.1.4 answers "how many real supercharges?"; Table I.1.5 answers "what rotates them?"; and Table I.1.6 answers "what field-theory data and tools come with the row?" The rows run from two to ten dimensions, the range in which pure supersymmetric Yang–Mills field theories exist without coupling to gravity. The $2d$ through $6d$ rows carry the main fixed-point structures; the $7d$ through $10d$ rows are the high-dimensional sixteen-supercharge SYM interface, including the F-theory 7-brane row at $8d$. The $32Q$ ten- and eleven-dimensional algebras are the gravity-coupled ceiling, not ordinary pure-field-theory rows.

dim	labels used here	real count	reading note
$2d$	$(0,1)/(1,0)$; $(0,2)/(1,1)$; $(2,2), (4,4), (8,8)$	$1, 2, 4, 8, 16$	(p, q) has $p + q$ real supercharges; chirality is part of the label.
$3d$	$\mathcal{N}$=$1, \ldots, 8$	$Q = 2\mathcal{N}$	The interacting superconformal list stops at $\mathcal{N} \leq 8$.
$4d$	$\mathcal{N}$=$1, 2, 3, 4$	$Q = 4\mathcal{N}$	$\mathcal{N} = 3$ has the same count as twelve real supercharges, not an extra reduction rung.
$5d$	$\mathcal{N}$=$1, 2$	$Q = 8, 16$	The sixteen-supercharge row is maximal SYM, not a 5d SCFT row.
$6d$	$(1,0), (1,1), (2,0)$	$Q = 8, 16, 16$	The two sixteen-supercharge labels differ by chirality and field content.
$7d$	$\mathcal{N}$=1	$Q = 16$	The 7d SYM row used by the G_2 roadmap interface.
$8d$	$\mathcal{N}$=1	$Q = 16$	The gauge-theory row on F-theory 7-branes.
$9d$	$\mathcal{N}$=1	$Q = 16$	Pure SYM exists as a non-gravitational theory.
$10d$	$\mathcal{N}$=1	$Q = 16$	The parent pure-SYM row; 32Q type-II algebras are gravity-coupled.

Table I.1.4: Master count table: quick lookup from each dimension-dependent SUSY label to its real-supercharge count. The ambient $32Q$ ceiling is tracked in Table I.1.1 and (I.1.4), not as a decoupled-QFT row.

dim	label	R-symmetry	caution
$2d$	$(0,1)/(1,0)$; $(0,2)$; $(2,2)$; $(4,4)$	none; $U(1)_R$; $U(1)_V \times U(1)_A$; $SU(2) \times SU(2)$, respectively	$U(1)_L$ in $(0,2)$ is flavor, not the superconformal R-current.
$3d$	$\mathcal{N}$=1; $\mathcal{N}$=2; $\mathcal{N} \geq 3$	none; $U(1)_R$; Spin($\mathcal{N}$), respectively	Maximal SYM has UV scalar-rotation Spin(7); the IR superconformal algebra has Spin$(8)_R$.
$4d$	$\mathcal{N}$=1; $\mathcal{N}$=2; $\mathcal{N}$=3; $\mathcal{N}$=4	$U(1)_R$; $SU(2)_R \times U(1)_R$; $SU(3) \times U(1)$; $SU(4)_R$, respectively	The $\mathcal{N} = 2$ row is $SU(2)_R \times U(1)_R$, not $SU(3)$.
$5d$	$\mathcal{N}$=1; $\mathcal{N}$=2	$SU(2)_R$; $USp(4)_R$, respectively	$U(1)_I/E_n$ is flavor, not R-symmetry; the $16Q$ row is maximal SYM, not a $5d$ SC algebra.
$6d$	$(1,0)$; $(1,1)$; $(2,0)$	$SU(2)_R$; Spin$(4)_R$; $USp(4)_R$=Spin$(5)_R$, respectively	The two $16Q$ labels have different chirality and different R-symmetry.
$7d$	$\mathcal{N}$=1	$SU(2)_R$	This is a sixteen-supercharge SYM row, not a separate fixed-point family.
$8d$	$\mathcal{N}$=1	$U(1)_R$	The $U(1)_R$ is the $SO(2)$ transverse rotation of the 7-brane gauge theory.
$9d$	$\mathcal{N}$=1	none continuous	The $SO(1)$ transverse rotation of the SYM parent is trivial.
$10d$	$\mathcal{N}$=1	none continuous	The parent pure-SYM row has no transverse rotation R-symmetry.

Table I.1.5: The master R-symmetry table. It isolates the automorphism group that rotates the supercharges and the flavor-versus-R distinctions that later exact computations depend on. Within each row, the entries in the label column and the R-symmetry column are read in the same order. The two-dimensional entries shown are those used in later chapters. Further chiral variants appear in the $2d$ chapter; this table is not a full classification of $2d$ superconformal symmetry.

dim	multiplets and branches	protected tools	later owners
$2d$	chiral, twisted-chiral, vector, Fermi; Higgs, Coulomb, LG phases	c-extremization for $(0,2)+$; elliptic genus	IV, VI.10, XII
$3d$	scalar, chiral, vector, hyper; Higgs, Coulomb; mirror pairs	F-maximization for $3d$ $\mathcal{N}\geq 2$ SCFTs with an R-current; S^3 partition function; $S^2\times S^1$ index; $\mathcal{N}>8$ free	VII, VIII, XIV
$4d$	chiral, vector, hyper; Higgs and Coulomb branches; Coulomb-branch low-energy data	a-maximization; superconformal index	V, VI, IX, XI, XIII
$5d$	vector, hyper; real Coulomb branch	5d index only for $\mathcal{N}=1$; no $\mathcal{N}=2$ SC algebra	X
$6d$	tensor, hyper, vector; tensor branch	anomaly 8-form and Green–Schwarz factorization	XII, XIII
$7d$	$\mathcal{N}=1$ vector multiplet; Coulomb branch	sixteen-supercharge SYM count	G_2 roadmap row
$8d$	$\mathcal{N}=1$ vector multiplet; two adjoint scalars	7-brane gauge theory; no SC extremization	X-A (F-theory)
$9d$	$\mathcal{N}=1$ vector multiplet; one adjoint scalar	reduction from $10d$ SYM; no SC extremization	high-dimensional SYM interface
$10d$	$\mathcal{N}=1$ vector multiplet	parent pure SYM; no SC extremization	II / high-dimensional SYM interface

Table I.1.6: The master table of field-theory tools. It records the field-theory objects, protected computations, and later volume homes associated with each dimension after the count and R-symmetry have already been fixed.

What this chapter states but does not prove.

The algebra fixes the counts and shortening conditions, while the dynamics determines which representations and states a theory realizes. It states two classification theorems and cites them rather than proving them. The first is the Nahm classification of super-Poincaré algebras, with its $d\leq 11$ and 32-supercharge ceiling (§I.1.1). The second is the classification of superconformal algebras, with its $d\leq 6$ interacting ceiling (§I.1.5). No later volume re-proves these, so they are cited classifications, not derivations supplied here. The chapter also states the BPS bound and its shortening, but computes no specific BPS spectrum. Which states saturate it, and how the central charge varies over a moduli space, is the dynamics studied in the dimension chapters and later constructions. Finally, the chapter does not derive supersymmetry preservation from external construction data, and it does not develop the Euclidean-spinor and localization-contour conventions that later index and partition-function chapters need; those are deferred to the first localization-using volume. The exit capability is this chapter's: align the supersymmetry notation through the master tables, read off the

R-symmetry, central-charge, and BPS roles, and translate a field-theory SUSY label into its real-supercharge count.

Exit checklist. After this chapter the reader can

1. translate any field-theory supersymmetry label in the master tables into a real supercharge count;
2. follow the $4Q$, $8Q$, and $16Q$ dimensional ladders under ordinary circle reduction;
3. identify the R-symmetry, central-charge, and BPS role attached to a row;
4. use the $SU(2)/SU(3)$ embedding example to decide whether the standard D-brane orbifold probe has $8Q$ / $4d$ $\mathcal{N}=2$ or $4Q$ / $4d$ $\mathcal{N}=1$;
5. state which classification results are being cited rather than proved.

Status and sources

The dimensional ladder and master tables use the Lorentzian spinor and real-supercharge conventions introduced in this chapter.

§I.1.1 The super-Poincaré algebra. For the displayed adjoint anticommutator (I.1.1), positivity follows in a unitary representation by $\langle\psi|\{Q,Q^\dagger\}|\psi\rangle = \|Q\psi\|^2 + \|Q^\dagger\psi\|^2 \geq 0$ [MATH]; this elementary implication is proved here. The existence and uniqueness of the graded extension under nontrivial-scattering hypotheses, and the conventional spin-≤ 2 ceiling, are imported classification results. The sources are Coleman and Mandula (1967) for the bosonic no-go, Haag et al. (1975) for the graded extension, and Nahm (1978) for the classification and ceiling. The scalar central term (I.1.2) is algebraic once that extension is assumed; the existence of a charged state is a separate physical question.

§I.1.2 The minimal spinor and the count. For finite-dimensional real Clifford modules in the displayed Lorentzian convention, the reality types in (I.1.4) are a mathematical classification [MATH]. Its proof is external, in Kugo and Townsend (1983), and is not reproduced here. The chapter verifies the representative gamma-matrix relations and then uses real Bott periodicity: explicit representatives $d = 2, \ldots, 9$ determine the eight residue classes; $d = 10, 11$ inherit the reality types of $d = 2, 3$. Multiplying each minimal real dimension by the stated multiplicity proves the count formula (I.1.3) within those hypotheses. The calculation does not establish which spinors or supercharges a particular physical theory realizes.

§I.1.2 R-symmetry. An automorphism commuting with Lorentz symmetry acts on the supercharges as in (I.1.5); this is the algebraic definition. The entries of Table I.1.5 also assign currents to physical theories, including the $2d\,(0,2)$ $U(1)_R$ versus left-moving $U(1)_L$ flavor current, the $5d$ $SU(2)_R$ versus $U(1)_I$ flavor current, and the UV/IR distinction for maximal $3d$ SYM. These assignments are physical inputs. Group rank and dimension comparisons do not derive current algebras or enhancement.

§I.1.3 The dimensional ladder. Given a supersymmetry-preserving untwisted reduction without a projection, restriction of each higher-dimensional supercharge gives the same real count in lower dimensions. The count identities in Table I.1.1 and (I.1.6)–(I.1.8) are conditional

arithmetic [MATH]; supersymmetry preservation is a physical premise. The displayed on-shell equality $d-2=\frac{1}{2}Q_{\min}$ in (I.1.9) is a degree-of-freedom calculation, not by itself an existence theorem for interacting SYM. The count does not establish the full $10d$-to-$2d$ reduction or derive the separate $6d\,(2,0)$ chiral branch. The D3-orbifold example uses the displayed $SU(4)$ spinor branching; applying that branching to a D3 probe requires the orbifold projection of Douglas and Moore (1996) for $\mathbb{C}^2/\Gamma$ and Douglas, Greene, et al. (1997) for $\mathbb{C}^3/\Gamma$. The representation arithmetic is conditional on those physical premises. Reducing the ten-dimensional gauge-field components gives $10-D$ scalars; identifying their rotations as a theory's UV R-symmetry is a further physical claim. The classical SYM reduction and its $\mathrm{Spin}(10-D)$ symmetry are described in Section 2 of Seiberg (1998); its Section 7 treats the distinct chiral six-dimensional tensor theory (called $(0,2)$ there, with the opposite chirality-label convention). Section 4 explains the three-dimensional UV Spin(7) symmetry and the Spin(8) symmetry required if the infrared limit is superconformal. The infrared existence and superconformality assumptions remain physical inputs.

§I.1.4 Central charges and the BPS bound. For the Hermitian block (I.1.10) in a unitary representation, $\det A=M^2-|Z|^2$ and the eigenvalues $M\pm|Z|$ prove the conditional bound (I.1.11) and the rank-one kernel at saturation [MATH]. This kernel calculation does not establish a BPS spectrum. The normalization of (I.1.12) is imported from Seiberg and Witten (1994): the block entry is $\sqrt{2}$ times its Z. The conditional 2^r oscillator count in the text explains the 16 versus 4 states for fixed charge and spin-zero Clifford vacuum; Section 4 of the same source gives this long/short comparison. It is not computed by the one-block kernel exercise. Within a short representation, the preserved combination annihilates a saturating state; existence and stability of such a one-particle state remain physical claims. Section 4 of Seiberg and Witten (1994) supports the decay condition at a wall of marginal stability. No chamber-independent degeneracy is claimed.

§I.1.5 Superconformal algebras and the unitarity bound. Assume a unitary $4d$ $\mathcal{N}=1$ SCFT, its actual superconformal R-current, a scalar primary annihilated by S, and the chosen Q/S anticommutator (I.1.15). Positivity of the descendant norm gives (I.1.16); a chiral primary saturates the branch and obeys (I.1.17) [MATH]. This is the conditional argument displayed in the chapter, with representation-theoretic context in Minwalla (1998); it is not a derivation of the full superconformal closure or the current assignment. The cubic calculation $3R(\Phi)=2$, followed by $\Delta=\frac{3}{2}R=1$, is conditional arithmetic. The calculation does not establish superconformal closure or the full radial-quantization positivity argument. The finite-dimensional superconformal classification and its $5d$ $F(4)$ row are imported from Nahm (1978). For current selection, Intriligator and Wecht (2003) supplies $4d$ $\mathcal{N}=1$ a-maximization, Jafferis (2012) the S^3 partition-function extremum for $3d$ $\mathcal{N}\geq 2$ theories with an R-current (subject to its accidental-symmetry caveat), and Benini and Bobev (2013) $2d$ $(0,2)$ c-extremization. The distinct $S^2\times S^1$ superconformal index is supported by Kapustin and Willett (2011). These imported physical principles are not proved by the cubic exercise.

§I.1.6 The master tables. Tables I.1.4–I.1.6 map the selected labels and later chapter homes. The count column inherits the stated spinor and reduction premises; the multiplets, protected tools, and later homes are navigation, not a single theorem.

Evidence scope. Here [MATH] labels only the explicitly conditional Clifford-module classification and algebraic implications just named. The other classifications and physical premises are imported from the cited sources.

Further reading

For the super-Poincaré algebra and the uniqueness theorems, standard textbook treatments are Weinberg (2013) and Terning (2006) and the review Sohnius (1985); the founding papers are Golfand and Likhtman (1971), Volkov and Akulov (1973), and Wess and Zumino (1974). Central charges and the BPS bound originate with Witten and Olive (1978). Compact pedagogical introductions at the level of these notes are Martin (1998), Bilal (2001), Lykken (1996), and Argyres (2001), and the superspace apparatus is catalogued in Gates et al. (1983).

Modern treatments of superconformal algebras and their representations across dimensions are given in Eberhardt (2021); the geometry of square-zero supercharges and the resulting classification of twists are developed in Eager et al. (2021) and Elliott et al. (2022). A recent computation-oriented survey of large-N limits across $3d$, $4d$, and $5d$ supersymmetric field theories is Santilli (2025).

References

Argyres, P. C. (2001). "An Introduction to Global Supersymmetry". In.

Benini, F. and N. Bobev (2013). "Exact two-dimensional superconformal R-symmetry and c-extremization". In: *Phys. Rev. Lett.* 110, p. 061601. arXiv: 1211.4030 [hep-th].

Bilal, A. (Jan. 2001). "Introduction to supersymmetry". In: arXiv: hep-th/0101055.

Coleman, S. R. and J. Mandula (1967). "All Possible Symmetries of the S Matrix". In: *Phys. Rev.* 159, pp. 1251–1256. DOI: 10.1103/PhysRev.159.1251.

Douglas, M. R., B. R. Greene, and D. R. Morrison (1997). "Orbifold resolution by D-branes". In: *Nucl. Phys. B* 506, pp. 84–106. DOI: 10.1016/S0550-3213(97)00517-8. arXiv: hep-th/9704151.

Douglas, M. R. and G. W. Moore (1996). "D-branes, quivers, and ALE instantons". In: arXiv preprint; not published in a journal. arXiv: hep-th/9603167.

Eager, R., I. Saberi, and J. Walcher (2021). "Nilpotence varieties". In: *Annales Henri Poincare* 22.4, pp. 1319–1376. DOI: 10.1007/s00023-020-01007-y. arXiv: 1807.03766 [hep-th].

Eberhardt, L. (2021). "Superconformal symmetry and representations". In: *J. Phys. A* 54.6, p. 063002. DOI: 10.1088/1751-8121/abd7b3. arXiv: 2006.13280 [hep-th].

Elliott, C., P. Safronov, and B. R. Williams (2022). "A taxonomy of twists of supersymmetric Yang–Mills theory". In: *Selecta Math.* 28.4, p. 73. DOI: 10.1007/s00029-022-00786-y. arXiv: 2002.10517 [math-ph].

Gates, S. J. et al. (1983). *Superspace Or One Thousand and One Lessons in Supersymmetry*. Vol. 58. Frontiers in Physics. arXiv: hep-th/0108200.

Golfand, Y. A. and E. P. Likhtman (1971). "Extension of the Algebra of Poincare Group Generators and Violation of p Invariance". In: *JETP Lett.* 13. Ed. by A. Salam and E. Sezgin, pp. 323–326. DOI: 10.1142/9789814542340_0001.

Haag, R., J. T. Lopuszanski, and M. Sohnius (1975). "All Possible Generators of Supersymmetries of the S Matrix". In: *Nucl. Phys. B* 88, p. 257. DOI: 10.1016/0550-3213(75)90279-5.

Intriligator, K. A. and B. Wecht (2003). "The Exact superconformal R symmetry maximizes a". In: *Nucl. Phys. B* 667, pp. 183–200. DOI: 10.1016/S0550-3213(03)00459-0. arXiv: hep-th/0304128.

Jafferis, D. L. (2012). "The Exact Superconformal R-Symmetry Extremizes Z". In: *JHEP* 05, p. 159. DOI: 10.1007/JHEP05(2012)159. arXiv: 1012.3210 [hep-th].

Kapustin, A. and B. Willett (2011). "Generalized Superconformal Index for Three Dimensional Field Theories". In: arXiv: 1106.2484 [hep-th].

Kugo, T. and P. K. Townsend (1983). "Supersymmetry and the Division Algebras". In: *Nucl. Phys. B* 221, pp. 357–380. DOI: 10.1016/0550-3213(83)90584-9.

Lykken, J. D. (June 1996). "Introduction to supersymmetry". In: *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 85–153. arXiv: hep-th/9612114.

Martin, S. P. (1998). "A Supersymmetry primer". In: *Adv. Ser. Direct. High Energy Phys.* 18. Ed. by G. L. Kane, pp. 1–98. DOI: 10.1142/9789812839657_0001. arXiv: hep-ph/9709356.

Minwalla, S. (1998). "Restrictions imposed by superconformal invariance on quantum field theories". In: *Adv. Theor. Math. Phys.* 2, pp. 783–851. DOI: 10.4310/ATMP.1998.v2.n4.a4. arXiv: hep-th/9712074.

Nahm, W. (1978). "Supersymmetries and their Representations". In: *Nucl. Phys. B* 135, pp. 149–166. DOI: 10.1016/0550-3213(78)90218-3.

Santilli, L. (2025). "Large N limits of supersymmetric quantum field theories: A pedagogical overview". In: *Fortschr. Phys.* 73.6, e70006. DOI: 10.1002/prop.70006. arXiv: 2501.05794 [hep-th].

Seiberg, N. (1998). "Notes on theories with 16 supercharges". In: *Nucl. Phys. B Proc. Suppl.* 67, pp. 158–171. DOI: 10.1016/S0920-5632(98)00128-5. arXiv: hep-th/9705117.

Seiberg, N. and E. Witten (1994). "Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426. [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

Sohnius, M. F. (1985). "Introducing Supersymmetry". In: *Phys. Rept.* 128, pp. 39–204. DOI: 10.1016/0370-1573(85)90023-7.

Terning, J. (2006). *Modern supersymmetry: Dynamics and duality*. DOI: 10.1093/acprof:oso/9780198567639.001.0001.

Volkov, D. V. and V. P. Akulov (1973). "Is the Neutrino a Goldstone Particle?" In: *Phys. Lett. B* 46, pp. 109–110. DOI: 10.1016/0370-2693(73)90490-5.

Weinberg, S. (June 2013). *The quantum theory of fields. Vol. 3: Supersymmetry*. Cambridge University Press.

Wess, J. and B. Zumino (1974). "Supergauge Transformations in Four-Dimensions". In: *Nucl. Phys. B* 70. Ed. by A. Salam and E. Sezgin, pp. 39–50. DOI: 10.1016/0550-3213(74)90355-1.

Witten, E. and D. I. Olive (1978). "Supersymmetry Algebras That Include Topological Charges". In: *Phys. Lett. B* 78, pp. 97–101. DOI: 10.1016/0370-2693(78)90357-X.

Chapter I.2

Common SUSY-QFT grammar: multiplets, local terms, vacua, anomalies

Reader contract. A field may sit in a chiral multiplet, but that name alone does not tell us which interactions it admits or which vacua it produces. The supercharges and R-symmetry fixed in Chapter 1 specify the algebra; the multiplets and local terms supply a theory in which that algebra acts. To find a vacuum we then solve the local equations, identify gauge-equivalent solutions, and ask which operators and anomalies persist beyond a chosen Lagrangian frame.

The same sequence works as a reading guide when a convenient Lagrangian is absent. We develop it through small calculations: eliminating an auxiliary field, taking a gauge quotient, constructing a ring of vacua, and tracing fermion charges. Each example fixes a shared physical question that the later dimension chapters answer with their own local tools.

What this chapter delivers

A local supersymmetric description has three linked parts: its multiplets and allowed terms, its vacuum equations and gauge equivalences, and its protected operators and symmetry responses. We calculate each part in a small example and show how it changes when the spacetime dimension changes. The detailed tools then belong to the dimension chapters: E/J systems and GLSMs live in $2d$, Chern–Simons terms and monopoles in $3d$, a-maximization and Seiberg-type technology in $4d$, prepotentials and instanton operators in $5d$, tensor-branch and Green–Schwarz anomaly technology in $6d$.

I.2.1 Multiplets are representations with local descendants

A supersymmetric field theory begins with an ordinary spacetime and a super-Poincaré algebra, but the fields do not appear one at a time. They appear in representations of the algebra. We call such a representation a *multiplet*. The word is intentionally neutral: a multiplet can be off-shell in a convenient formalism, on-shell in a more economical one, or non-Lagrangian if the theory is known by protected data rather than by a field list. What survives across all these cases is that supersymmetry relates bosons and fermions and packages their local descendants into a common object.

The smallest useful distinction is between three pieces of data.

1. The *spacetime dimension* and the amount of supersymmetry, fixed by Chapter 1.
2. The *multiplet type*, such as chiral, vector, hyper, tensor, or Fermi.
3. The *local terms* allowed by supersymmetry: kinetic terms, superpotential-type terms, moment-map or D-type terms, and in special dimensions topological couplings.

For example, a four-dimensional $\mathcal{N} = 1$ chiral multiplet is not the same representation as a two-dimensional $(0, 2)$ chiral multiplet, even though the same word "chiral" is used. The label is meaningful only after the algebra and dimension have been fixed.

Table I.2.1 is not a classification theorem. It is a reading aid. A dimension chapter may refine a row, split it, or replace it by a more natural language. The reading rule is that a multiplet name is not a promise of a universal component formula. It is a promise that the later chapter will state which algebra is acting and what local data the multiplet carries.

A minimal superspace slogan

Superspace is a powerful notation, but we will use only its slogan here. One introduces anticommuting coordinates θ so that some supersymmetry transformations become translations in θ. A constraint on a superfield then defines a multiplet, and an integral over part or all of the θ-space defines a supersymmetric local term.

For a four-dimensional $\mathcal{N} = 1$ Lagrangian, the familiar schematic form is

$$\mathcal{L} = \int d^4\theta\, K(\Phi^\dagger e^V, \Phi) + \left(\int d^2\theta\, W(\Phi) + \int d^2\theta\, \frac{\tau}{16\pi i}\, \mathrm{Tr}\, W^\alpha W_\alpha + \text{c.c.} \right). \qquad \text{(I.2.1)}$$

Here Φ denotes chiral multiplets, V vector multiplets, K a Kähler potential, W a holomorphic superpotential, and τ a complexified gauge coupling. The two integration measures in Equation (I.2.1) distinguish full-superspace D-type terms from chiral-superspace F-type terms. In particular, the same chiral measure carries both the superpotential and the holomorphic gauge kinetic coupling; their component equations play different physical roles.

Multiplet	Typical use	Book-level rule
Chiral	complex scalar plus fermions in $4d$ $\mathcal{N}=1$, and related lower-dimensional systems	Carries holomorphic local data. Its detailed component content is dimension-dependent.
Vector	gauge field plus superpartners	Carries gauge redundancy and a moment-map/D-type constraint when a Lagrangian description exists; a $3d$ $\mathcal{N}=2$ vector also contains a real scalar after reduction.
Hyper	extended-SUSY matter	Usually best viewed as paired chiral data from a smaller-SUSY perspective; its quaternionic structure is dimension-specific.
Tensor	higher-form field strength plus partners	Essential in $6d$. Tensor branches and Green–Schwarz terms are owned by the $6d$ chapter.
Fermi	left-moving fermionic multiplet in $2d$ $(0,2)$ language	Carries E- and J-type data. Here we name the words; the $2d$ chapter does the real work.
Current multiplet	conserved currents and stress tensor	Encodes global symmetries, R-currents, anomalies, and protected observables.

Table I.2.1: Multiplet types and the book-level reading rule. The table is a reading aid, not a classification theorem.

The available measures and multiplets change with dimension. In $2d$ $(0,2)$, the natural language uses chiral and Fermi multiplets with E- and J-functions. In $3d$, Chern–Simons couplings and monopole operators become central. In $5d$, a cubic prepotential controls the Coulomb branch. In $6d$, tensor multiplets and anomaly polynomials organize the story. The common question is which local term the chosen superalgebra permits and what equation or protected coupling it produces.

Off-shell convenience versus physical content

The reader will often meet phrases such as "off-shell multiplet" and "auxiliary field". We use them in a pragmatic way. An off-shell description realizes the supersymmetry transformations without imposing the equations of motion, usually by adding auxiliary fields. This can make local terms compact to write and vacuum equations mechanically visible. It is not, by itself, the physical definition of the theory.

The Wess–Zumino example in Equation (I.2.3) has an auxiliary field F. Before eliminating it, the bosonic part contains

$$|F|^2 + F\,\frac{\partial W}{\partial \phi} + \overline{F}\,\frac{\partial \overline{W}}{\partial \overline{\phi}}. \tag{I.2.2}$$

The equation of motion sets $F = -\partial\overline{W}/\partial\overline{\phi}$, equivalently $\overline{F} = -\partial W/\partial\phi$, and the positive potential $V_F = |\partial W|^2$ follows. The auxiliary field makes the positive potential and

its zero locus visible algebraically. It has no propagating degree of freedom. Indeed, writing the displayed terms as $|F+\overline{W_\phi}|^2-|W_\phi|^2$ shows both the stationary value of F and the sign of the potential.

This distinction becomes useful later because different dimensions have different off-shell technologies. Four-dimensional $\mathcal{N}=1$ superspace is economical. Extended supersymmetry can force one to use a smaller visible subalgebra, harmonic or projective superspace, or no compact off-shell formalism at all. Volume 0 will not standardize those choices. It will standardize the questions: what are the multiplets, what local terms are allowed, which auxiliary or protected equations follow, and what physical data remain after quotienting redundancies?

Smaller-SUSY language is often a chart

A common move is to describe a theory with a smaller visible supersymmetry than the theory actually has. A four-dimensional $\mathcal{N}=2$ vector multiplet can be described, in $\mathcal{N}=1$ language, by a vector multiplet plus an adjoint chiral multiplet. A hypermultiplet can be described by a pair of chiral multiplets in conjugate representations. This is a chart on the local grammar, not a statement that the extra supersymmetry has disappeared.

The same idea appears after dimensional reduction. A component that was a gauge field in a higher dimension can become a scalar in a lower dimension. The multiplet name then changes with the algebra. This is why the volume keeps Chapter 1 and the present bridge separate: Chapter 1 says how the supercharges and R-symmetries reduce; this chapter says how the local objects are named once a dimension and visible algebra have been chosen.

I.2.2 F-terms, D-terms, and the first vacuum equations

In a Lagrangian frame, supersymmetric vacua are often found by eliminating auxiliary fields. This is the most useful entry point for the later chapters, because it turns a local term into an equation. We record the minimal pattern.

Consider a single four-dimensional $\mathcal{N}=1$ chiral multiplet Φ with scalar component ϕ and superpotential

$$W(\Phi)=\frac{m}{2}\Phi^2+\frac{\lambda}{3}\Phi^3. \tag{I.2.3}$$

The auxiliary field F is eliminated by

$$F=-\frac{\partial\overline{W}}{\partial\bar{\phi}}, \qquad V_F(\phi)=\left|\frac{\partial W}{\partial\phi}\right|^2=\left|m\phi+\lambda\phi^2\right|^2. \tag{I.2.4}$$

A supersymmetric vacuum in this sector solves

$$\frac{\partial W}{\partial\phi}=0. \tag{I.2.5}$$

If $m\lambda \neq 0$, the equation has two isolated solutions, $\phi = 0$ and $\phi = -m/\lambda$. The potential vanishes at both, so the holomorphic derivative identifies the supersymmetric vacua in this ungauged example. An *F-flat equation* is the vanishing of that derivative, or of the corresponding local equation in another dimension.

For a gauge theory with chiral fields ϕ_i of charges q_i under a U(1), a D-type auxiliary field gives

$$D = \sum_i q_i |\phi_i|^2 - \zeta, \qquad V_D = \frac{g^2}{2} D^2. \tag{I.2.6}$$

Here ζ is a Fayet–Iliopoulos parameter when the dimension and supersymmetry allow it. The D-flat equation is

$$\sum_i q_i |\phi_i|^2 = \zeta. \tag{I.2.7}$$

For a general compact gauge group the same statement becomes a moment-map equation

$$\mu(\phi) = \zeta_{\text{central}}. \tag{I.2.8}$$

The level ζ_{central} must lie in the gauge-invariant central part of the Lie algebra dual. For a semisimple nonabelian factor the corresponding level is zero. Thus one often writes $\mu^a = 0$ for nonabelian generators and $\mu_{U(1)} = \zeta$ for abelian trace factors. The exact normalization of μ depends on conventions. The invariant content is that D-type terms impose real moment-map equations before one divides by gauge equivalence.

Common misconception: F-terms and D-terms are specific local terms

"F-term" and "D-term" do not mean "everything holomorphic" and "everything non-holomorphic" in a dimension-independent way. They mean specific supersymmetric local terms in a specified algebra. We use the four-dimensional $\mathcal{N} = 1$ notation only as a compact example, not as a universal formalism.

The quotient is part of the equation

Solving the equations is not yet the vacuum space. Gauge-equivalent solutions represent the same physical vacuum. A basic U(1) example makes the point. Let Q and $\widetilde{Q}$ have charges $+1$ and -1, with no superpotential and $\zeta = 0$. The D-flat equation is

$$|Q|^2 - |\widetilde{Q}|^2 = 0. \tag{I.2.9}$$

Modulo the U(1) action

$$(Q, \widetilde{Q}) \mapsto (e^{i\alpha} Q, e^{-i\alpha} \widetilde{Q}), \tag{I.2.10}$$

the gauge-invariant coordinate is

$$M = Q\widetilde{Q}. \tag{I.2.11}$$

For the complexified action, a monomial $Q^a \widetilde{Q}^b$, with $a, b \geq 0$, transforms by λ^{a-b} for $\lambda \in \mathbb{C}^*$. An invariant polynomial therefore contains only monomials with $a = b$, each equal to M^a. Thus the polynomial invariant ring is $\mathbb{C}[Q, \widetilde{Q}]^{\mathbb{C}^*} = \mathbb{C}[M]$. The D-flat condition fixes the difference of the two magnitudes, while the gauge action removes

their relative phase. Their product retains the remaining complex magnitude and phase, so the quotient is parameterized by $M \in \mathbb{C}$. At $M = 0$ both fields vanish. The coordinate is obtained only after imposing the equation and dividing by the gauge action.

A slightly different U(1) example explains why the sign of a real parameter can change the description. Let X_1 and X_2 both have charge $+1$ and take $\zeta > 0$. The D-flat equation

$$|X_1|^2 + |X_2|^2 = \zeta \tag{I.2.12}$$

modulo the common phase gives $\mathbb{P}^1$. At $\zeta = 0$, the quotient collapses to the origin in the gauge-invariant affine description. The later $2d$ GLSM chapter will make this kind of statement precise using its own language. Here we only keep the reading rule: an FI-like parameter can separate phases, and a quotient by gauge equivalence is part of the vacuum construction.

Complex quotient language

There is an equivalent algebraic way to package many of these examples. Instead of solving the real moment-map equation and dividing by the compact gauge group, one often solves the holomorphic F-flat equations and divides by the complexified gauge group, with a stability condition determined by the FI parameter. For the X_1, X_2 example with $\zeta > 0$, the complexified group is $\mathbb{C}^*$ acting by

$$(X_1, X_2) \mapsto (\lambda X_1, \lambda X_2), \qquad \lambda \in \mathbb{C}^*. \tag{I.2.13}$$

The stable locus removes the origin because it cannot satisfy the positive D level $\zeta > 0$. The resulting quotient is

$$(\mathbb{C}^2 - \{0\})/\mathbb{C}^* \simeq \mathbb{P}^1. \tag{I.2.14}$$

This is the same physical quotient as the D-flat sphere modulo U(1). The bridge does not develop geometric invariant theory. It only records why the same vacuum may be written in a differential form, as a moment-map quotient, or in an algebraic form, as equations modulo a complexified redundancy.

The distinction matters because the later chapters use both languages. A $2d$ GLSM phase is usually clearest in quotient language. A $3d$ Coulomb branch may require quantum coordinates that are not visible in a naive classical quotient. A $4d$ chiral ring is often presented algebraically, while its unitary vacuum interpretation remembers D-flatness. These are not three different theories. They are three controlled coordinate systems on the same local problem when the assumptions of the chosen frame apply.

A superpotential ideal

The simplest F-term ideal already shows how relations enter a protected ring. Take three chiral multiplets X, Y, Z and

$$W = XYZ. \tag{I.2.15}$$

The F-flat equations are

$$YZ = 0, \qquad XZ = 0, \qquad XY = 0. \tag{I.2.16}$$

The classical chiral ring in this frame is

$$\mathcal{R}_{XYZ} = \mathbb{C}[X, Y, Z]/(XY, XZ, YZ). \tag{I.2.17}$$

Any point on one axis makes two fields vanish, so every pairwise product vanishes. Conversely, the three relations force at most one of X, Y, Z to be nonzero at a point. The vacuum set is therefore the union of three coordinate axes meeting at the origin. The relations reveal both the separate components and their singular intersection.

I.2.3 Branches, protected rings, and local coordinates

Supersymmetric vacua often form families. A connected component or irreducible piece of such a family is called a *branch*. The name of a branch is physical shorthand, not a theorem by itself. A Higgs branch is usually visible in a frame where charged matter has expectation values. A Coulomb branch is usually visible in a frame where vector-multiplet scalars, dual photons, or related abelianized data parameterize the vacuum. A mixed branch contains both kinds of data.

The word "usually" matters. Dualities can exchange descriptions, and strong-coupling physics can introduce coordinates not visible in a weakly coupled Lagrangian. A branch label therefore records how the branch is seen in a chosen frame. It is not an invariant name that every dual frame must use in the same way.

Label	**Typical weak-frame signal**	**Reading rule**
Higgs branch	charged matter expectation values and gauge symmetry broken partly or fully	Use the label when matter coordinates or gauge-invariant composites are the natural local coordinates.
Coulomb branch	vector-multiplet scalars, abelianization, dual photons, monopole or instanton coordinates in suitable dimensions	Use the label only after the dimension-specific chapter states the appropriate coordinates.
Mixed branch	simultaneous matter and vector-multiplet coordinates	Treat as an intersection or fibration when the later chapter supplies the local structure.
Conformal manifold	exactly marginal deformations modulo redundancies	We name it; the dimension chapter supplies the protected marginality tests.

Table I.2.2: Branch labels and their frame-dependent weak-coupling signals.

Protected chiral or BPS operator rings are a second way to describe vacuum data. In a Lagrangian example, one begins with gauge-invariant polynomials in fields and

divides by relations generated by F-flat equations. If

$$\mathcal{R} = \mathbb{C}[x, y, z]/(xy), \tag{I.2.18}$$

then the relation $xy = 0$ says that the affine variety is the union of two coordinate planes meeting along the z-axis. A coarse Hilbert series is

$$H_{\mathcal{R}}(t) = \frac{1 - t^2}{(1 - t)^3}. \tag{I.2.19}$$

This formula is included only to normalize the vocabulary. Volume VI uses Hilbert-series and perfect-matching technology in a sharper way, and later parts of the project use protected rings as computational tools. Volume 0 only needs the reader to recognize that a ring presentation consists of generators, relations, and gradings.

What a Hilbert series is allowed to mean here

A Hilbert series is a counting device for graded operators. In the toy ring $\mathcal{R} = \mathbb{C}[x, y, z]/(xy)$, assign $\deg x = \deg y = \deg z = 1$. The polynomial ring has generating function $(1 - t)^{-3}$. Multiplication by xy injects the degree-shifted polynomial ring into $\mathbb{C}[x, y, z]$, so the ideal contributes $t^2/(1 - t)^3$. Subtracting it gives Equation (I.2.19). That is the entire level of detail needed in this chapter.

The later chapters will be more careful. A physically meaningful Hilbert series may refine the grading by flavor fugacities, impose gauge invariance through an integral or a Molien sum, or include quantum coordinates. It may also fail to capture the full geometry if the chosen ring is not the right protected object. The displayed $(1 - t^2)/(1 - t)^3$ form follows from the single nonzero-divisor relation xy; it does not apply unchanged to an arbitrary ideal. The resulting coefficients $1, 3, 5, 7, \ldots$ count independent operators at successive degrees in this ring.

> **Common misconception: not every vacuum component is a Higgs branch**
>
> The chapter does not identify every vacuum component with a Higgs branch. A Higgs/Coulomb/mixed label depends on which multiplets provide the coordinates in the chosen description, and in some dimensions the natural Coulomb coordinates are intrinsically quantum.

When no Lagrangian is available

Some supersymmetric field theories are not best introduced by a weakly coupled Lagrangian. The same questions still apply. A theory can be specified, partly or wholly, by protected data:

$$\mathfrak{D}_{\text{protected}} = \left\{ \begin{array}{l} \text{superalgebra and R-symmetry,} \\ \text{global symmetry and anomaly data,} \\ \text{protected operators and rings,} \\ \text{deformations, defects, and boundary data} \end{array} \right\}. \tag{I.2.20}$$

This list is not a replacement for dynamics. It is a way to read the later chapters without assuming that every supersymmetric QFT has a convenient elementary-field Lagrangian. In particular, non-Lagrangian four-dimensional $\mathcal{N} = 2$ theories and six-dimensional SCFTs are not exceptions to the grammar; they are cases where the grammar is applied to protected observables and deformation data rather than to a microscopic component action.

There is a discipline to this. Calling a theory non-Lagrangian should not license vague prose. One should still be able to say which supersymmetry algebra is realized, which global symmetries act on protected operators, which deformations are allowed, which anomalies are known, and what evidence supports an identification with another description. A later chapter may use a curve, a category of defects, a BPS spectrum, or an anomaly polynomial as primary data. These can provide field-theoretic data once their symmetry and observable content is specified.

I.2.4 Symmetries, currents, and anomaly traces

Supersymmetric theories come with several kinds of symmetries. The most important distinction is between gauge redundancy and global symmetry. A gauge symmetry is not an ordinary physical symmetry acting faithfully on the Hilbert space; it is a redundancy in the description. A global symmetry acts on operators and states. An R-symmetry is a global symmetry that also acts on the supercharges. A flavor symmetry commutes with the supercharges.

This distinction controls anomaly language. A gauge anomaly is a consistency obstruction for a dynamical gauge symmetry unless it is cancelled, for example by a local counterterm, by anomaly inflow, or by a Green–Schwarz-type mechanism when such a mechanism exists in that dimension. A 't Hooft anomaly of a global symmetry is protected response data: it must match across dual descriptions and becomes an obstruction only if one tries to gauge that global symmetry. An R-symmetry anomaly can be both a diagnostic and, in special dimensions, part of an extremization principle.

For a four-dimensional chiral fermion in representation ρ of a symmetry group, the perturbative cubic anomaly is measured schematically by

$$\mathcal{A}^{abc} = \mathrm{Tr}_{\text{left fermions}}\, \rho(T^a)\{\rho(T^b), \rho(T^c)\}. \tag{I.2.21}$$

For a mixed R-flavor-flavor anomaly one similarly traces the R-charge of the fermion times the two flavor generators:

$$\mathcal{A}_{RFF} = \mathrm{Tr}_{\text{left fermions}}\, R_\psi\, \rho(T^a)\rho(T^b). \tag{I.2.22}$$

The subscript ψ is deliberate. In a four-dimensional $\mathcal{N} = 1$ chiral multiplet, if the scalar has R-charge R_ϕ, then the fermion has R-charge

$$R_\psi = R_\phi - 1. \tag{I.2.23}$$

Anomaly traces are fermion traces. Forgetting this shift is one of the most common ways to get the wrong answer.

Gauge cancellation in a vectorlike fixture

As a minimal four-dimensional fixture, take an $\mathrm{SU}(N_c)$ gauge theory with a chiral multiplet Q in the fundamental representation and a chiral multiplet $\widetilde{Q}$ in the antifundamental. The left-handed fermion in Q contributes the cubic gauge anomaly coefficient $A(\mathbf{N})$, while the left-handed fermion in $\widetilde{Q}$ contributes $A(\overline{\mathbf{N}}) = -A(\mathbf{N})$. The pair is therefore vectorlike with respect to the gauge group:

$$A(\mathbf{N}) + A(\overline{\mathbf{N}}) = 0. \tag{I.2.24}$$

For conjugate generators $T^a_{\overline{\mathbf{N}}} = -(T^a_{\mathbf{N}})^T$, the cubic symmetric trace gains three minus signs, giving the stated opposite coefficients. For $\mathrm{SU}(2)$ the perturbative cubic coefficient of each representation vanishes separately. This is not a statement about the flavor anomalies of the same fields. The flavor symmetry can still have nonzero 't Hooft anomalies, and those anomalies are meaningful infrared data. The same fermions participate in both computations; what changes is whether the traced generators belong to a gauge redundancy or to a global symmetry.

This fixture also explains why the word "matter" is too imprecise for an anomaly computation. Scalars contribute to the multiplet and to the vacuum coordinates, but perturbative chiral anomalies are fermion traces. Gauge representations, chirality, and R-charge shifts have to be read at the fermion level.

> **Common misconception: gauge anomalies versus 't Hooft anomalies**
>
> A gauge anomaly and a 't Hooft anomaly of a global symmetry play different roles. Gauge anomalies must be removed for the description to be consistent. 't Hooft anomalies are protected response data; they are allowed because they are observable constraints, and they must match across dual descriptions.

A minimal anomaly-matching fixture

Suppose two descriptions claim to flow to the same infrared theory. If they have the same non-anomalous global symmetry G_F, then the 't Hooft anomalies for G_F and for any preserved R-symmetry must match. In a four-dimensional $\mathcal{N} = 1$ frame the two basic fermion anomaly traces are

$$\mathrm{Tr}\, R = \sum_\psi R_\psi, \qquad \mathrm{Tr}\, R^3 = \sum_\psi R_\psi^3. \tag{I.2.25}$$

The trial central charge built from these traces is

$$a_{\mathrm{trial}}(R) = \frac{3}{32}\left(3\,\mathrm{Tr}\, R^3 - \mathrm{Tr}\, R\right). \tag{I.2.26}$$

This equation is here only as vocabulary. The actual a-maximization principle belongs to the $4d$ chapter, where four-dimensional dynamics is the subject. We record the ingredients: a candidate R-symmetry, a list of fermions, and anomaly traces. One caution is worth stating now. Matching anomalies is a necessary consistency check on a

proposed duality, not a proof of it. Two theories can share all their 't Hooft anomalies and still be different; the dimension chapters supply the further evidence a duality claim requires.

The same caution applies in other dimensions. Two-dimensional gravitational and flavor anomalies are naturally chiral and lead into c-extremization in $(0, 2)$ theories. Three-dimensional parity anomalies and contact terms are tied to Chern–Simons levels. Six-dimensional theories organize anomalies in an eight-form polynomial, often with tensor-branch Green–Schwarz cancellation. These are not variants of Equation (I.2.21) with the dimension label changed. They are distinct technologies whose shared concept is anomaly as protected local or global data.

Common misconception: the R-symmetry is not "whatever U(1) is convenient"

R-symmetry is not "whatever U(1) is convenient". It is the symmetry that acts on the supercharges, and in interacting theories the exact infrared R-symmetry can mix with flavor symmetries. We name this fact; the dimension chapter performs the relevant extremization or replacement principle.

Background fields as a unifying language

A clean way to remember the gauge/global distinction is to couple global symmetries to nondynamical background fields. If the background field is made dynamical, the symmetry has become a gauge redundancy and the anomaly must be cancelled. If the field remains a background, the anomaly is a well-defined response of the quantum theory. This response can be encoded as a phase of the partition function, a contact term, or an anomaly polynomial, depending on dimension.

This language lets the same bridge sentence cover several later cases without pretending they are identical. In $2d$, a chiral imbalance of left-moving and right-moving fermions produces gravitational and flavor anomalies. In $3d$, integrating out massive fermions can shift Chern–Simons contact terms, and the parity anomaly constrains allowed backgrounds and levels. In $4d$, cubic and mixed 't Hooft anomalies constrain dual descriptions and feed R-symmetry tests. In $6d$, the anomaly polynomial and Green–Schwarz terms are central data. The common statement is only this: anomalies are local or global obstructions/responses attached to background or dynamical symmetry fields, and the dimension chapter tells which mathematical object represents them.

I.2.5 Deformations and parameters

The same local grammar also organizes deformations of a theory. At a specified fixed point, a deformation by an operator of scaling dimension Δ is *relevant* when $\Delta < d$, *marginal* when $\Delta = d$, and *irrelevant* when $\Delta > d$. An *exactly marginal* coupling lies along a conformal manifold after quantum effects and redundancies are taken into account. A *mass parameter* is a coefficient of a mass deformation; when a suitable flavor symmetry and supersymmetric background multiplet exist, it can be packaged as a background value. A *coupling* is a coefficient of a local term. An *FI parameter*, when available, is a parameter in a moment-map equation.

We use this vocabulary without asserting that the same parameter exists in every dimension. This is especially important for supersymmetric field theories in dimensions above four, where Yang–Mills couplings are dimensionful. With the standard normalization $(1/g^2)\int F^2$, the coupling has $[g^2] = 4 - d$, while the inverse coefficient has $[1/g^2] = d - 4$. Thus the existence of a UV fixed point is a dynamical question, not a consequence of the local term alone. It is also important in two and three dimensions, where real masses, FI terms, twisted masses, Chern–Simons levels, and monopole operators have their own protected roles.

Word	**Bridge meaning**	**Where details live**
Superpotential deformation	Holomorphic local deformation of chiral data	$2d$, $3d$, and $4d$ chapters in their own notation
Mass parameter	Background value associated to a flavor symmetry	Dimension chapters, especially $3d$ and $4d$ examples
FI parameter	Moment-map parameter when the algebra allows it	$2d$ GLSMs and $3d$ gauge dynamics
Exactly marginal coupling	Protected marginal deformation modulo redundancies	$4d$ conformal-manifold discussion
Topological level	Quantized coefficient such as a Chern–Simons level	$3d$ chapter
Tensor-branch parameter	Scalar expectation value in tensor multiplet data	$6d$ chapter

Table I.2.3: Deformation and parameter vocabulary, and the chapters that own the technical details.

Common misconception: a deformation is not specified by its name alone

A deformation is not specified by its name alone. One must say which operator is being added, which symmetry protects or forbids it, what its dimension is, and whether the parameter is continuous, quantized, or background-field data.

Relevant, marginal, and protected are different adjectives

We keep three adjectives separate. A deformation is *relevant*, *marginal*, or *irrelevant* according to the scaling dimension of the operator at the fixed point under discussion. It is *supersymmetric* if it can be written as an allowed supersymmetric local term. It is *protected* if supersymmetry and other symmetries constrain its quantum behavior strongly enough that a finite computation or index captures it.

These adjectives often correlate, but they are not synonyms. A superpotential mass term can be relevant and supersymmetric. A gauge kinetic term may be marginal in four dimensions but dimensionful in five. A Chern–Simons level in three dimensions is supersymmetric and quantized, but it is not a continuous exactly marginal parameter.

A tensor-branch coefficient in six dimensions is not a four-dimensional gauge coupling in disguise.

The operational rule is to ask four questions:

1. What operator or background multiplet is being turned on?
2. What symmetries, including R-symmetry, allow the term?
3. What is the mass dimension of the parameter at the point being studied?
4. Is the parameter continuous, discrete, quantized, or a background choice?

This rule prevents the bridge from importing a four-dimensional intuition into every dimension.

Masses as background multiplet data

Mass parameters are a useful place to see the local/global vocabulary at work. A flavor symmetry current lives in a current multiplet. Coupling that current multiplet to a nondynamical background vector multiplet and giving the background scalar a value produces a mass parameter, when the dimension and supersymmetry allow such a coupling. In a weak four-dimensional $\mathcal{N} = 1$ frame, a holomorphic mass can appear in a superpotential term

$$W_{\text{mass}} = m\, Q\widetilde{Q}. \tag{I.2.27}$$

In lower dimensions, real masses and twisted masses are often more natural. This chapter does not equate these. It records the shared logic: a mass is not merely a number in a Lagrangian; it is usually tied to a flavor symmetry, a background multiplet, and a choice of preserved supersymmetry.

This statement has dimension-specific realizations. Holomorphic masses in a four-dimensional $\mathcal{N} = 1$ frame are naturally superpotential or chiral-spurion data, whereas real masses in three dimensions and twisted masses in two dimensions are often background-vector-multiplet scalar data. We record the shared flavor-symmetry organization without identifying these mechanisms.

This viewpoint is especially useful when comparing two descriptions. If a duality maps a flavor symmetry G_F on one side to the same symmetry on the other, its mass deformations should map as background values for G_F. If an FI parameter is present, it should map according to the topological or gauge symmetry structure of that dimension. The details are dimension-specific, but the elementary sentence is stable: deformations are organized by the multiplets and symmetries to which their parameters belong.

I.2.6 The routing table for the dimension chapters

The Wess–Zumino derivative and the quotient examples answered two common questions, but a new dimension changes which multiplets and protected coordinates can answer them. Table I.2.4 identifies the additional tools needed in each case.

The examples differ because each dimension has characteristic operators and couplings. In $2d$ a Fermi multiplet brings E and J data into the vacuum problem. In $3d$

Dim.	Owns	This chapter says
$2d$	$(2,2)$ and $(0,2)$ multiplets, GLSMs, E/J data, elliptic genera, c-extremization	Chiral, vector, and Fermi multiplets are local representations; E and J are dimension-specific local data.
$3d$	Chern–Simons terms, parity anomaly, monopole operators, mirror symmetry, F-maximization	Topological terms and disorder operators are part of the local/protected grammar in $3d$.
$4d$	$\mathcal{N}=1$ chiral/vector grammar, Seiberg duality, a-maximization, $\mathcal{N}=2$ Seiberg–Witten data, class S	F-terms, D-terms, chiral rings, and anomaly traces have a familiar compact notation.
$5d$	Coulomb-branch prepotentials, instanton particles/operators, enhanced flavor symmetry, $5d$ SCFT limits	Couplings and Coulomb coordinates are dimensionful and can signal UV fixed-point data.
$6d$	tensor branches, anomaly polynomials, Green–Schwarz cancellation, BPS strings	Tensor multiplets and anomaly data are essential; this chapter does not compute the eight-form.
$7d$–$10d$	high-dimensional supersymmetric field theories used as boundary/interface or reduction inputs	The common words still apply, but these chapters are compact and field-theory-first.

Table I.2.4: The routing table: which dimension chapter owns which technology, and what the bridge alone fixes.

monopole operators may supply protected coordinates absent from a polynomial ring of elementary fields. In $4d$ the same fermion traces used above constrain exact R-symmetry and proposed dualities. In $5d$ a prepotential and instanton particles enter the Coulomb description, while $6d$ tensor branches require anomaly-polynomial and Green–Schwarz data. The shared questions remain useful precisely because their answers change.

This is also the reason for the title "supersymmetric field theories" rather than a gauge-centered title. Gauge theories provide many of the most explicit frames, but they are not the whole subject. The dimension chapters must be able to discuss sigma models, Landau–Ginzburg models, non-Lagrangian SCFTs, boundary/interface systems, and compact higher-dimensional field theories without pretending that every example is best introduced by a gauge field and charged matter.

Common misconception: non-Lagrangian does not mean non-field-theoretic

Non-Lagrangian does not mean non-field-theoretic. A theory can be defined or constrained by symmetry, anomalies, protected rings, defects, and deformation data even when no weakly coupled elementary Lagrangian is the right starting point.

Near-misses that the bridge forbids

This chapter is as much about preventing false identifications as about introducing terms. A four-dimensional F-term and a two-dimensional $(0,2)$ J-term are related by family resemblance, but they are not the same object with a new name. A three-dimensional monopole operator is a protected disorder operator, not merely a scalar expectation value that was forgotten in the elementary field list. A five-dimensional instanton particle is tied to the topological current and UV completion data, not to the four-dimensional instanton correction story by a change of notation. The six-dimensional tensor branch has its own anomaly-cancellation mechanism, not a D-term quotient written in a larger font.

The appropriate first question is therefore specific to the object. For a local term, identify the available supersymmetric measure or protected coupling. For a vacuum, identify both its equations and the equivalence relation. For an anomaly, decide whether the symmetry is dynamical or background and which anomaly object exists in that dimension. These choices determine which calculation can follow; a change of dimension is not merely a change of notation.

I.2.7 Theory cards: the reading format

For later use it is helpful to compress the shared vocabulary into a standard "theory card". This is not a mathematical definition of QFT. It is a reading format. When a dimension chapter introduces a theory $\mathcal{T}$, the first pass should identify

$$\mathcal{T}: \quad (d, \mathcal{Q}, G_R;\ \mathfrak{M}_{\text{local}}; \mathcal{I}_{\text{terms}}; \mathcal{V}_{\text{vac}}; \mathfrak{D}_{\text{protected}}). \tag{I.2.28}$$

Here d is the spacetime dimension, $\mathcal{Q}$ the supersymmetry algebra data from Chapter 1, G_R the R-symmetry, $\mathfrak{M}_{\text{local}}$ the multiplet or protected local-data list, $\mathcal{I}_{\text{terms}}$ the allowed local interactions, $\mathcal{V}_{\text{vac}}$ the visible vacuum equations and quotient, and $\mathfrak{D}_{\text{protected}}$ the protected data used for comparison.

The card format keeps examples honest. If a theory is Lagrangian, the local-data entry may be a literal field list. If a theory is non-Lagrangian, the local-data entry may instead be a protected operator spectrum, an anomaly package, or a deformation pattern. If a theory has a gauge-theory frame in one regime, the card records that frame without declaring it to be the unique definition of the theory. If two descriptions are dual, their cards need not have identical local-data entries, but their protected data and deformation responses must be compatible.

Card entry	Question	Examples of acceptable answers
Algebra	Which SUSY algebra and R-symmetry?	$4d$ $\mathcal{N}=1$ with $\mathrm{U}(1)_R$; $2d$ $(0,2)$ with $\mathrm{U}(1)_R$; $6d$ $(1,0)$ with $\mathrm{SU}(2)_R$.
Local data	What carries the fields or protected observables?	Chiral and vector multiplets; Fermi multiplets; tensor multiplets; current multiplets; protected operator data.
Local terms	What interactions are allowed here?	Superpotential, D-term, FI term, Chern–Simons level, prepotential term, Green–Schwarz coupling, each only in its proper dimension.
Vacua	What equations and equivalences are visible?	F-flat equations, moment-map equations, complex quotient, quantum Coulomb coordinates, branch decomposition.
Symmetries	Which symmetries are redundancies or physical?	Gauge group, flavor group, topological symmetry, R-symmetry, higher-form symmetry when the chapter introduces it.
Protected data	What must match or be preserved?	Anomaly traces, protected rings, BPS spectra, indices, conformal-manifold data, defect or boundary data.

Table I.2.5: The theory-card reading format: the question each card entry answers.

A sample card

Return to the simple U(1) theory with Q and $\widetilde{Q}$ of charges $+1$ and -1, no superpotential, and $\zeta = 0$. Its elementary card is

$$\begin{aligned} &d, \mathcal{Q}: && \text{chosen from the dimension chapter,} \\ &\mathfrak{M}_{\text{local}}: && Q, \widetilde{Q} \text{ plus a U(1) vector multiplet,} \\ &\mathcal{I}_{\text{terms}}: && \text{kinetic terms, no } W, \zeta = 0, \\ &\mathcal{V}_{\text{vac}}: && |Q|^2 - |\widetilde{Q}|^2 = 0 \text{ modulo U(1),} \\ &\text{invariant coordinate}: && M = Q\widetilde{Q}. \end{aligned} \tag{I.2.29}$$

This card does not determine all dynamics. It does determine what the bridge is allowed to say: the vacuum quotient has a coordinate M, the branch label depends on the dimension and frame, and any anomaly statement must specify the fermions and symmetries being traced.

The same card can be refined in several directions. A two-dimensional version may add E/J data and GLSM phases. A three-dimensional version may add real masses, FI parameters, and monopole operators. A four-dimensional version may add anomaly matching and an exact R-symmetry problem. This chapter does not perform these

refinements, but it prevents the reader from losing track of which entry of the card is being modified.

I.2.8 Worked dictionary: one frame, many later uses

We close with a small dictionary calculation that will be reused in several dimensions. Take chiral fields Φ_i, a holomorphic superpotential $W(\Phi_i)$, and a gauge group G. In a Lagrangian frame, the local supersymmetric data determine the classical vacuum equations

$$\frac{\partial W}{\partial \Phi_i} = 0, \qquad \mu(\phi) = \zeta_{\text{central}}, \qquad \text{modulo } G. \tag{I.2.30}$$

The first equation is F-flatness, the second is D-flatness or moment-map flatness, with nonzero levels only in central or abelian directions; the last phrase is gauge equivalence. The protected ring in this frame is represented schematically by

$$\mathcal{R} = \frac{\mathbb{C}[\text{gauge-invariant chiral words}]}{(\partial W = 0)}. \tag{I.2.31}$$

Equation (I.2.31) is not a universal theorem. Quantum effects can deform it, monopole or instanton operators can add coordinates, and non-Lagrangian theories can require a different presentation. Its use is pedagogical: it identifies the ingredients that later chapters will keep modifying.

The same frame has candidate global symmetries G_F, candidate R-symmetries, and possible anomalies. To compute an anomaly one lists the fermions and traces their charges, not the scalar charges directly. To test a duality one compares protected data: symmetries, anomaly traces, operator maps, branch dimensions, and deformations. To study a family of vacua one decides which coordinates remain light and which equivalences have already been divided out.

Reading a new theory

When later chapters present a new supersymmetric field theory, read it in the following order. First identify the algebra: dimension, number of supercharges, and R-symmetry. Second identify the visible multiplets or, if the theory is non-Lagrangian, the protected data standing in for multiplets. Third list the local supersymmetric terms. Fourth derive the vacuum equations or protected-ring relations that are visible in the chosen frame. Fifth separate gauge redundancies from global symmetries. Sixth record anomalies and other protected quantities that have to survive along renormalization-group flow.

This order keeps distinct operations from being conflated. If one starts from a branch name before identifying the multiplets, one can call a quantum Coulomb coordinate a Higgs coordinate by accident. If one starts from scalar R-charges before shifting to fermion R-charges, the anomaly trace is wrong. If one writes a quotient before stating the stability condition or FI chamber, one may collapse the wrong locus. If one treats a 't Hooft anomaly of a global symmetry like a gauge anomaly, one may incorrectly discard physical data.

Table I.2.5 identifies the data; Table I.2.6 orders the questions needed to use them. For example, the symmetry role must be fixed before its anomaly can be interpreted, and the quotient must be specified before a vacuum coordinate is counted.

Checkpoint	**Action before moving on**	**Mistake it blocks**
Algebra fixed	Name the dimension, supercharges, and R-symmetry from Chapter 1.	Treating the same multiplet word as dimension-independent.
Local frame chosen	Say whether the data are elementary fields, protected data, or both.	Importing full superspace or a gauge-theory frame when not given.
Equations and quotient	Write the visible equations together with the quotient or stability condition.	Forgetting gauge equivalence or complexified stability.
Protected coordinates	State whether ring, index, BPS, monopole, or instanton data add coordinates.	Assuming all coordinates are elementary fields.
Symmetry role separated	Mark each symmetry as gauge, flavor, topological, or R before tracing anomalies.	Cancelling a 't Hooft anomaly of a global symmetry or matching a gauge redundancy.
Handoff chosen	Route to the dimension chapter only after the previous entries are fixed.	Running extremization, monopole, or tensor-branch machinery too early.

Table I.2.6: The handoff checklist for entering a dimension chapter. It is the operational use of Table I.2.5, not a second theory-card schema.

The card records a theory's available local and protected data; the checklist asks which of those data enter a proposed calculation. The answer will look different in $2d$, $3d$, $4d$, $5d$, $6d$, and $7d$–$10d$. The grammar is common; the physics is dimensional.

Exit checklist

The chapter has done its job if the reader can perform the following translation without importing dimension-specific machinery too early.

Prompt	Expected translation
“Write the F-flat equations.”	Differentiate the holomorphic local term in the appropriate multiplet language.
“Take the vacuum quotient.”	Solve the equations and divide by gauge equivalence, or describe the invariant coordinate ring.
“Name the branch.”	State which local coordinates define the branch in this frame; do not assume the label is duality-invariant.
“Check the anomaly.”	Trace over fermions, distinguish gauge consistency from global protected data, and use the dimension’s anomaly technology.
“Route the topic.”	Put E/J, GLSM, Chern–Simons, monopoles, extremization, Seiberg–Witten, prepotentials, tensor branches, and anomaly polynomials in their dimension chapters.

Status and sources

This chapter uses the spinor and real-supercharge conventions of Chapter 1.

§I.2.1 Multiplets. A multiplet is a representation of the super-Poincaré algebra; a multiplet name is dimension- and algebra-dependent, not a universal component formula (Table I.2.1). The four-dimensional chiral and vector entries use Wess and Bagger (1992); the two-dimensional $(0,2)$ Fermi entry uses Distler (1995); the six-dimensional tensor entry uses Bergshoeff et al. (1996).

§I.2.2 F-terms, D-terms, and vacua. The F-flat equation $\partial W/\partial\phi = 0$, the D-flat / moment-map equation $\mu(\phi) = \zeta_{\text{central}}$, and the gauge quotient, in the four-dimensional $\mathcal{N} = 1$ notation used as a compact example, with the $U(1)$ meson $M = Q\widetilde{Q}$ and the $\mathbb{C}^*$-quotient $\mathbb{P}^1$ (Wess and Bagger (1992) for the superspace grammar; Witten (1993) for the positive-FI quotient example). Under the displayed charges, $Q\widetilde{Q}$ is neutral and generates the invariant polynomial ring. [MATH] by the integer weight condition. The identification of the D-flat quotient with that affine coordinate is derived above. The two-charge-one positive-level $S^3/U(1) \simeq \mathbb{P}^1$ quotient is a separate conditional argument. [MATH]. The XYZ F-term ideal is given in (I.2.17). Computing the moduli space requires the gauge quotient as well as $F = 0$ and $D = 0$. The compact-group moment-map statement has a wider scope than the cited two-dimensional GLSM example.

§I.2.3 Branches and protected rings. A branch label is frame-dependent shorthand (Table I.2.2); a protected ring is generators modulo F-flat relations, counted by a graded Hilbert series at the dictionary level only (Benvenuti et al. (2010) for the Hilbert-series language; Gaiotto (2009) and Seiberg (1996) for examples beyond elementary Lagrangians). For the degree-one toy ring $\mathbb{C}[x,y,z]/(xy)$, multiplication by the nonzero polynomial xy is injective, so subtracting its shifted polynomial-ring series gives $(1-t^2)/(1-t)^3$. [MATH]. The ring-to-series implication follows from this multiplication map. The branch names and the schematic protected-data list are supported by selected four- and six-dimensional examples, not by a universal classification.

§I.2.4 Symmetries, currents, and anomaly traces. The gauge / 't Hooft distinction; anomaly traces are fermion traces with the $R_\psi = R_\phi - 1$ shift; the vectorlike cancellation $A(\mathbf{N}) + A(\overline{\mathbf{N}}) = 0$; the trace data $\operatorname{Tr} R, \operatorname{Tr} R^3$ and a_{trial} named as vocabulary only (Intriligator and Wecht (2003) for the a-maximization ingredients; Seiberg (1995) 't Hooft matching across duals). The two-fermion example illustrates the R-shift and trial-a arithmetic. For a conjugate pair of four-dimensional left-handed $\mathrm{SU}(N_c)$ representations, the cubic symmetric trace changes sign under conjugation, so the perturbative local coefficients sum to zero. [MATH] for that representation-theoretic implication; it says nothing about global anomalies. Gauge anomaly cancellation is a consistency condition. 't Hooft anomaly matching constrains a proposed duality.

§I.2.5 Deformations and parameters. Relevant / marginal / exactly-marginal / mass / coupling / FI vocabulary (Table I.2.3), with the dimensionful-coupling analysis $[g^2] = 4 - d$ flagging that a UV fixed point is dynamical. Given $[F] = 2$ and a dimensionless action, $[1/g^2] + 4 = d$, hence $[g^2] = 4 - d$. [MATH] for this dimensional implication only.

§§I.2.6–I.2.8 Routing, theory cards, and the reading sequence. The routing table (Table I.2.4) assigning E/J+GLSM+c-extremization to $2d$, Chern–Simons+monopoles+F-maximization to $3d$, a-maximization+Seiberg–Witten+class S to $4d$, prepotentials+instanton operators to $5d$, tensor branches+anomaly polynomials to $6d$; the theory-card format (Table I.2.5) and the reading sequence (Table I.2.6). These are reading and routing choices, not mathematical theorems (Gadde et al. (2014) a forward pointer for the $3d$ protected-data vocabulary). The theory card (I.2.28) also applies to theories without a conventional Lagrangian. The examples do not establish a general fixed-point deformation classification.

Further reading

The chiral, vector, and current multiplets and the superspace formalism are developed in Wess and Bagger (1992); the founding superfield papers are Salam and Strathdee (1975) and Ferrara et al. (1974). The supercurrent and the supersymmetry multiplets of conserved currents are treated in Komargodski and Seiberg (2010), and rigid supersymmetry on curved backgrounds in Festuccia and Seiberg (2011) and Dumitrescu (2017). Non-renormalization from holomorphy is explained in Seiberg (1993), the four-dimensional a-theorem constraining renormalization-group flows in Komargodski and Schwimmer (2011), and supersymmetry breaking as a general phenomenon in Luty (2005).

Modern extensions of the symmetry-and-anomaly dictionary to higher-form and higher-group symmetries are reviewed pedagogically in Gomes (2023) and Bhardwaj et al. (2024).

References

Benvenuti, S., A. Hanany, and N. Mekareeya (2010). "The Hilbert Series of the One Instanton Moduli Space". In: *JHEP* 06, p. 100. DOI: 10.1007/JHEP06(2010)100. arXiv: 1005.3026 [hep-th].

Bergshoeff, E., E. Sezgin, and E. Sokatchev (1996). "Couplings of self-dual tensor multiplet in six dimensions". In: *Class. Quant. Grav.* 13, pp. 2875–2886. arXiv: hep-th/9605087.

Bhardwaj, L. et al. (2024). "Lectures on generalized symmetries". In: *Phys. Rept.* 1051, pp. 1–87. DOI: 10.1016/j.physrep.2023.11.002. arXiv: 2307.07547 [hep-th].

Distler, J. (1995). *Notes on (0,2) Superconformal Field Theories*. arXiv: hep-th/9502012.

Dumitrescu, T. T. (2017). "An introduction to supersymmetric field theories in curved space". In: *J. Phys. A* 50.44, p. 443005. DOI: 10.1088/1751-8121/aa62f5. arXiv: 1608.02957 [hep-th].

Ferrara, S., J. Wess, and B. Zumino (1974). "Supergauge Multiplets and Superfields". In: *Phys. Lett. B* 51, p. 239. DOI: 10.1016/0370-2693(74)90283-4.

Festuccia, G. and N. Seiberg (2011). "Rigid Supersymmetric Theories in Curved Superspace". In: *JHEP* 06, p. 114. DOI: 10.1007/JHEP06(2011)114. arXiv: 1105.0689 [hep-th].

Gadde, A., S. Gukov, and P. Putrov (2014). "Walls, Lines, and Spectral Dualities in 3d Gauge Theories". In: *JHEP* 05, p. 047. DOI: 10.1007/JHEP05(2014)047. arXiv: 1302.0015 [hep-th].

Gaiotto, D. (2009). *N=2 dualities*. arXiv: 0904.2715 [hep-th].

Gomes, P. R. S. (2023). "An introduction to higher-form symmetries". In: *SciPost Phys. Lect. Notes* 74, p. 1. DOI: 10.21468/SciPostPhysLectNotes.74. arXiv: 2303.01817 [hep-th].

Intriligator, K. A. and B. Wecht (2003). "The Exact superconformal R symmetry maximizes a". In: *Nucl. Phys. B* 667, pp. 183–200. DOI: 10.1016/S0550-3213(03)00459-0. arXiv: hep-th/0304128.

Komargodski, Z. and A. Schwimmer (2011). "On Renormalization Group Flows in Four Dimensions". In: *JHEP* 12, p. 099. DOI: 10.1007/JHEP12(2011)099. arXiv: 1107.3987 [hep-th].

Komargodski, Z. and N. Seiberg (2010). "Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity". In: *JHEP* 07, p. 017. DOI: 10.1007/JHEP07(2010)017. arXiv: 1002.2228 [hep-th].

Luty, M. A. (Sept. 2005). "2004 TASI lectures on supersymmetry breaking". In: *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics in $D \geqq 4$*, pp. 495–582. arXiv: hep-th/0509029.

Salam, A. and J. A. Strathdee (1975). "On Superfields and Fermi-Bose Symmetry". In: *Phys. Rev. D* 11, pp. 1521–1535. DOI: 10.1103/PhysRevD.11.1521.

Seiberg, N. (1993). "Naturalness versus supersymmetric nonrenormalization theorems". In: *Phys. Lett. B* 318, pp. 469–475. DOI: 10.1016/0370-2693(93)91541-T. arXiv: hep-ph/9309335.

Seiberg, N. (1995). "Electric - magnetic duality in supersymmetric nonAbelian gauge theories". In: *Nucl. Phys. B* 435, pp. 129–146. DOI: 10.1016/0550-3213(94)00023-8. arXiv: hep-th/9411149.

Seiberg, N. (1996). *Non-trivial Fixed Points of The Renormalization Group in Six Dimensions*. arXiv: hep-th/9609161.

Wess, J. and J. Bagger (1992). *Supersymmetry and Supergravity*. 2nd ed. Princeton University Press.

Witten, E. (1993). "Phases of N=2 theories in two dimensions". In: *Nucl. Phys. B* 403, pp. 159–222. arXiv: hep-th/9301042.

Chapter I.3

$4d\ \mathcal{N} = 1$ supersymmetric field theories

The physical route. In four-dimensional supersymmetric gauge theory, holomorphy turns ultraviolet couplings into information about strongly coupled vacua. A mass threshold changes the power of the dynamical scale; the same scale then enters gaugino condensation and the quantum constraints of SQCD. Global symmetries give a second route into the infrared. Their fermion anomalies fix admissible R-symmetries, determine conformal central charges, and supply necessary checks of Seiberg duality. We develop these connections from the multiplet and vacuum language of Chapters 1 and 2. The calculations progress from scale matching and the chiral ring to pure super Yang–Mills, exact R-charges, a-maximization, the SQCD phases, and a rank-condition example of metastable supersymmetry breaking. The given dP_0 quiver illustrates how the same field-theory calculations apply when a D-brane construction supplies the matter content.

What this chapter delivers

The holomorphic data of a $4d\ \mathcal{N} = 1$ theory, the superpotential and the gauge coupling, and the dynamically generated scale Λ with its threshold matching (§I.3.1); the chiral ring and non-renormalization (§I.3.2); pure super Yang–Mills, gaugino condensation, and the Witten index (§I.3.3); the $U(1)_R$ symmetry, the NSVZ beta function, and the superconformal R (§I.3.4); 't Hooft anomaly matching and the Konishi anomaly as tools, with the SQCD traces run explicitly (§I.3.5); a-maximization, the prototype of the extremization through-line, with SQCD, adjoint SQCD, and the dP_0 quiver worked (§I.3.6); SQCD across the flavor window, the Affleck–Dine–Seiberg runaway, the quantum-deformed moduli space, s-confinement, and Seiberg duality with five explicit anomaly traces (§I.3.7); and the Witten-index criterion for dynamical supersymmetry breaking with a worked rank-condition example (§I.3.8).

I.3.1 Multiplets, holomorphy, and the dynamical scale

The number of supercharges explains which dimensional reductions can reuse this language. A $4d$ $\mathcal{N}=1$ theory has four real supercharges, as do $3d$ $\mathcal{N}=2$ after circle reduction and $2d$ $\mathcal{N}=(2,2)$ after torus reduction. In four dimensions its R-symmetry is $U(1)_R$. The reduced theories retain four real supercharges, while their local terms, anomalies, and extremization principles take dimension-specific forms.

A $4d$ $\mathcal{N}=1$ theory is built from two multiplets fixed in Chapter 2. A chiral multiplet Φ carries a complex scalar, a Weyl fermion, and an auxiliary field F. A vector multiplet V carries a gauge field, a gaugino, and an auxiliary field D. The F-terms and gauge kinetic terms are specified by holomorphic functions. The superpotential W depends on the chiral superfields, while the gauge coupling and theta angle combine into

$$\tau \;=\; \frac{\theta}{2\pi} \;+\; \frac{4\pi i}{g^2}. \tag{I.3.1}$$

The classical supersymmetric vacua solve the F-flat equations $F_i = \partial W/\partial\phi_i = 0$ and the D-flat (moment-map) equation $D=0$ modulo gauge equivalence, as in Chapter 2. For SQCD below the gauge group is $SU(N_c)$. The D-brane quiver example starts with $\prod_i U(N_i)$; we isolate its interacting $SU(N)$ factors when computing central charges.

Holomorphy makes the coupling's one-loop running especially useful. It produces a scale invariant under changes of renormalization point and relates the scales on opposite sides of a massive-flavor threshold. The vanishing of its one-loop coefficient will also mark the upper edge of the SQCD conformal window.

The one-loop coefficient and the dynamical scale. A non-abelian gauge theory is not scale invariant at one loop. Its running coupling is set by the one-loop coefficient

$$b_0 \;=\; 3\,T(\mathrm{adj}) \;-\; \sum_i T(r_i) \;=\; 3\,N_c \;-\; \sum_i T(r_i), \tag{I.3.2}$$

the gauge contribution $3T(\mathrm{adj}) = 3N_c$ for $SU(N_c)$ (the adjoint Dynkin index is $T(\mathrm{adj}) = N_c$ in the normalization $T(\square)=\frac{1}{2}$) minus the Dynkin indices $T(r_i)$ of the matter chiral multiplets. For $SU(N_c)$ with N_f fundamental *flavors* the matter is N_f quarks Q in the fundamental plus N_f antiquarks $\widetilde{Q}$ in the antifundamental, so $2N_f$ fundamental chiral fields, each of index $T(\square)=\frac{1}{2}$. The matter contributes $2N_f\cdot\frac{1}{2} = N_f$, and

$$b_0 \;=\; 3N_c - N_f. \tag{I.3.3}$$

The count is $3N_c - N_f$, not $3N_c - 2N_f$: it is the sum of Dynkin *indices* (a half each), not a count of fields. Getting this wrong by a factor of two on the matter is the most common arithmetic slip in the whole subject, and it moves the top of the conformal window. A few worked values fix the formula, each (I.3.2) read with the right Dynkin

indices,

$$b_0\big|_{SU(3),\,N_f=0} = 9, \qquad b_0\big|_{SU(3),\,N_f=6} = 9-6 = 3,$$
$$b_0\big|_{SU(3),\,N_f=9} = 0, \qquad b_0\big|_{\text{adj},\,N_f=0} = 3N_c - N_c = 2N_c,$$
$$b_0\big|_{\mathcal{N}=4} = 3N_c - 3N_c = 0. \tag{I.3.4}$$

The one-loop coefficient vanishes at $N_f = 9$. It also vanishes for $\mathcal{N} = 4$ (three adjoint chirals), where no holomorphic one-loop scale is generated. The stronger statement that $\mathcal{N} = 4$ super Yang–Mills is exactly conformal follows from the extended supersymmetry and is not derived here.

Dimensional transmutation now trades the dimensionless coupling for a dimensionful, renormalization-group-invariant, holomorphic scale. The one-loop running of the holomorphic coupling is exact in the holomorphic scheme; the Wilsonian superpotential has no perturbative renormalization. With the coefficient b_0 defined above,

$$\tau(\mu) \;=\; \tau(\mu_0) \;+\; \frac{b_0}{2\pi i}\,\ln\frac{\mu_0}{\mu}, \tag{I.3.5}$$

whose imaginary part gives the one-loop coupling relation in this normalization,

$$\frac{8\pi^2}{g^2(\mu)} \;=\; b_0\,\ln\frac{\mu}{\Lambda}. \tag{I.3.6}$$

We package this running into a single dimensionful object, written in the form that avoids a branch ambiguity,

$$\Lambda^{b_0} \;=\; \mu^{\,b_0}\, e^{\,2\pi i\,\tau(\mu)}, \tag{I.3.7}$$

with μ the sliding scale. (The equivalent $\Lambda = \mu\, e^{2\pi i\tau/b_0}$ is the same object once a b_0-th root is chosen; we keep the power form (I.3.7) in the main text because it is single-valued.) The combination is built so that the explicit μ dependence cancels the running of τ. Differentiating the logarithm of (I.3.7),

$$b_0\,\frac{d\ln\Lambda}{d\ln\mu} \;=\; b_0 \;+\; 2\pi i\,\frac{d\tau}{d\ln\mu} \;=\; b_0 \;+\; 2\pi i\cdot\left(-\frac{b_0}{2\pi i}\right) \;=\; 0, \tag{I.3.8}$$

where the explicit μ^{b_0} contributes $+b_0$ and the running (I.3.5), which has $d\tau/d\ln\mu = -b_0/2\pi i$, contributes $-b_0$. So $d\ln\Lambda/d\ln\mu = 0$: along the one-loop trajectory Λ does not move. This Λ is the renormalization-group-invariant strong-coupling scale, the variable in which gaugino condensation, the Affleck–Dine–Seiberg superpotential, and Seiberg duality are all written. It is genuinely nonperturbative data: $\Lambda \sim \mu\, e^{-8\pi^2/(b_0 g^2)}$ is exponentially small in $1/g^2$ and invisible to any finite order of perturbation theory.

At the upper edge $N_f = 3N_c$ the coefficient b_0 vanishes. The one-loop holomorphic scale construction (I.3.7) then degenerates ($\Lambda^0 = 1$ carries no scale). This marks the asymptotic-freedom boundary of §I.3.7; it does not justify assigning a finite infrared confinement scale to that theory.

Threshold matching: integrating out one flavor. The holomorphic scale obeys a simple matching rule when a heavy flavor is removed, and that rule is a clean first

computation. Take $SU(N_c)$ with N_f flavors and give the last flavor a mass by adding $\Delta W = m\, Q_{N_f}\widetilde{Q}_{N_f}$ to the superpotential. Below the mass m the heavy quarks decouple and the theory is $SU(N_c)$ with $N_f - 1$ light flavors, governed by its own scale Λ_{N_f-1}. Holomorphy of the gauge coupling, with the threshold at $\mu = m$, demands that the two holomorphic scales agree across the threshold:

$$\Lambda_{N_f-1}^{\,3N_c-(N_f-1)} \;=\; m\,\Lambda_{N_f}^{\,3N_c-N_f}. \tag{I.3.9}$$

Read the exponents,

$$b_0^{\rm lo} = 3N_c - (N_f-1) = b_0^{\rm hi} + 1, \qquad b_0^{\rm hi} = 3N_c - N_f, \tag{I.3.10}$$

so the exponent shifts up by exactly one, and that one is exactly $2T(\square) = 2\cdot\frac{1}{2} = 1$, the total Dynkin index of the removed pair $(Q_{N_f}, \widetilde{Q}_{N_f})$. The factor of m on the right carries the one extra power of mass dimension that the higher exponent on the left needs, so the equation is dimensionally homogeneous: $[\Lambda_{N_f-1}^{b_0^{\rm hi}+1}] = [\Lambda_{N_f-1}]^{b_0^{\rm hi}+1}$ matches $[m][\Lambda_{N_f}]^{b_0^{\rm hi}}$. The relation (I.3.9) is what makes Λ a scale relation one can carry through the entire flavor window: removing a flavor raises the exponent by the removed index and inserts a factor of the mass.

Worked numbers. Take $N_c = 3$. The $N_f = 6$ theory has $b_0^{\rm hi} = 9 - 6 = 3$, and after integrating out one flavor the $N_f = 5$ theory has $b_0^{\rm lo} = 9 - 5 = 4$. The matching (I.3.9) reads

$$\Lambda_5^4 \;=\; m\,\Lambda_6^3 \quad\Longrightarrow\quad \Lambda_5 \;=\; \left(m\,\Lambda_6^3\right)^{1/4}, \tag{I.3.11}$$

a clean fourth root. As the mass m is raised toward the ultraviolet, $\Lambda_5 \to \infty$ relative to Λ_6, the statement that removing matter makes a theory more strongly coupled at a fixed scale. Iterating down to $N_f = 0$ (all flavors massive) recovers pure $SU(3)$ super Yang–Mills with $b_0 = 9$,

$$\Lambda_{\rm SYM}^9 \;=\; m_1\cdots m_6\,\Lambda_6^3, \tag{I.3.12}$$

the matching that will tie the SQCD scale to the gaugino condensate of §I.3.3.

If one assumes equal scales $\Lambda_{N_f-1} = \Lambda_{N_f} \equiv \Lambda$ without matching at the massive threshold, (I.3.9) would require

$$\Lambda^{\,3N_c-(N_f-1)} \;=\; m\,\Lambda^{\,3N_c-N_f} \quad\Longleftrightarrow\quad m = \Lambda. \tag{I.3.13}$$

Both sides have the same mass dimension. The equal-scale assumption fails for generic $m \neq \Lambda$, although it happens to satisfy the matching equation at $m = \Lambda$. The threshold mass cannot be omitted from the general relation. This threshold rule is the $4d$ $\mathcal{N} = 1$ analog of the scale-matching the $4d$ $\mathcal{N} = 2$ chapter performs for its own (different) scale, and the two should never be conflated: that chapter's scale runs with an $\mathcal{N} = 2$ one-loop coefficient and is matched to this one only across an $\mathcal{N} = 1$ mass deformation, as §I.3.3 explains.

Common misconception: the gauge algebra fixes the theory

The Lagrangian fixes only the gauge *algebra*. The global form of the gauge *group* is extra data. The algebra $\mathfrak{su}(2)$ is realized by $SU(2)$ and by $SO(3) = SU(2)/\mathbb{Z}_2$, and these are *different* quantum theories: they admit different spectra of genuine line operators (which Wilson and 't Hooft lines are present), equivalently a different discrete one-form symmetry and different allowed 't Hooft fluxes. Two theories with the same algebra and the same local action can be globally distinct. The classification of global forms by genuine line operators is developed in Part III. The Lagrangian, by itself, does not fix the theory.

I.3.2 The chiral ring and non-renormalization

The protected backbone of a $4d$ $\mathcal{N} = 1$ theory is its chiral ring. A chiral operator is one annihilated by all the $\bar{Q}$ supercharges. Two chiral operators that differ by a $\bar{Q}$-exact piece have the same correlators in any supersymmetric configuration, so the chiral operators are counted modulo $\bar{Q}$-exact terms. The product of two chiral operators is again chiral, and the product is commutative in the ring (the operators can be separated and their ordering is $\bar{Q}$-exact). The chiral ring is therefore a commutative ring of protected operators, graded by the abelian charges. Classically, in a tree-level Lagrangian frame its F-term quotient is the ring of gauge-invariant chiral words modulo the ideal generated by $\partial W = 0$, exactly the schematic $\mathcal{R} = \mathbb{C}[\text{chiral words}]/(\partial W)$ of Chapter 2. Quantum dynamics can deform these relations, as at $N_f = N_c$ below.

Generators: mesons and baryons. For $SU(N_c)$ SQCD with N_f flavors the gauge-invariant chiral generators are built by contracting color indices. The meson is the simplest,

$$M^{i}{}_{j} \;=\; Q^{i}{}_{a}\,\widetilde{Q}^{a}{}_{j}, \tag{I.3.14}$$

an $N_f \times N_f$ matrix in flavor indices i, j with the color index a contracted. When the ranks allow it ($N_f \geq N_c$) one can also antisymmetrize N_c quarks on their color indices with an epsilon tensor to form baryons,

$$B^{i_1 \cdots i_{N_c}} \;=\; \epsilon^{a_1 \cdots a_{N_c}}\, Q^{i_1}_{\;\;a_1} \cdots Q^{i_{N_c}}_{\;\;a_{N_c}}, \tag{I.3.15}$$

and an analogous $\widetilde{B}$ from the antiquarks. These generators are graded by the global charges; with $B(Q) = +1$, $B(\widetilde{Q}) = -1$ and the R-charges adding from the constituents,

$$\begin{aligned} &B(M) = 0, \quad B(B) = N_c, \quad B(\widetilde{B}) = -N_c, \\ &R(M) = 2R(Q) = 2\big(1 - \tfrac{N_c}{N_f}\big), \quad R(B) = N_c R(Q). \end{aligned} \tag{I.3.16}$$

The classical chiral ring is the polynomial ring in $M, B, \widetilde{B}$ modulo the relations forced by the epsilon identities. These relations are not optional: they encode the geometry of the classical moduli space. A clean example is $N_f = N_c$, where the single baryon and antibaryon and the $N_c \times N_c$ meson obey

$$\det M - B\widetilde{B} \;=\; 0, \tag{I.3.17}$$

the statement that the meson, baryon, and antibaryon are not independent: a single holomorphic relation cuts the chiral ring down to the moduli space of a rank-N_c matrix with its baryonic completion. We will see this relation deform quantum-mechanically in §I.3.7.

A dimension count makes the chiral ring concrete. Consider $SU(N_c)$ with $N_f \geq N_c$ flavors on its classical Higgs branch. The scalars are $2N_fN_c$ complex quark components; the D-flat condition and the gauge quotient remove $\dim SU(N_c) = N_c^2 - 1$ complex directions (the broken generators), so the generic Higgs branch has complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}_{\text{Higgs}} \;=\; 2N_fN_c - (N_c^2 - 1), \tag{I.3.18}$$

and the gauge-invariant generators (the mesons, with the baryons when $N_f \geq N_c$) coordinatize exactly this space modulo their classical relations. At $N_f = N_c$, for instance, the two counts agree,

$$\underbrace{2N_c^2 - (N_c^2 - 1)}_{\text{moduli-space dim}} = N_c^2 + 1 = \quad \underbrace{N_c^2 + 2 - 1}_{M\,+\,B,\,\widetilde{B}\,-\,\text{relation}} \quad , \tag{I.3.19}$$

the $N_f^2 = N_c^2$ meson components plus the two baryons minus the one relation (I.3.17). The chiral ring and the moduli-space dimension agree, the cross-check the later quiver calculations use. For the $\mathbb{C}^3/\mathbb{Z}_3$ quiver of §I.3.6 the chiral generators are instead the gauge-invariant closed loops of bifundamentals, $\mathrm{Tr}\,(X_{01}^a X_{12}^b X_{20}^c)$, the cubic mesonic operators whose ring is the coordinate ring of the singular cone the D-brane probes.

Non-renormalization. Holomorphy plus the broken flavor symmetries give the non-renormalization theorems. The Wilsonian superpotential receives no perturbative corrections: treating each coupling as a background chiral spurion and demanding holomorphy in those spurions, together with the flavor and R-symmetries and the weak-coupling limit, forbids any perturbative shift of W. The holomorphic gauge coupling runs only at one loop in a holomorphic scheme. The physical, canonically normalized coupling does run to all orders, but its running is fixed exactly by the NSVZ formula of §I.3.4, not by an independent series. These are statements we use throughout; their proofs are developed in Part III with the rest of the SQCD dynamics. The lesson to carry forward is that the holomorphic data, W and τ, are rigid. The resulting F-term relations make the chiral ring a computable description of protected holomorphic operators.

I.3.3 Pure super Yang–Mills: gaugino condensation and the index

The simplest $4d$ $\mathcal{N} = 1$ dynamics has no matter at all: pure $SU(N_c)$ super Yang–Mills, just the vector multiplet. It is the cleanest place to see Λ, the R-symmetry, and the index work together, and it is the worked instance the breaking criterion of §I.3.8 rests on.

The classical R-symmetry and its anomaly. Classically pure SYM has a $U(1)_R$ acting on the gaugino λ, which has R-charge one (the field-strength superfield W_α has

R-charge one, the gauge-kinetic chiral superfield $\mathrm{Tr}\, W_\alpha W^\alpha$ has R-charge two). The Adler–Bell–Jackiw (ABJ) anomaly breaks this continuous $U(1)_R$ to a discrete subgroup. The anomalous divergence of the R-current is proportional to $2T(\mathrm{adj}) = 2N_c$ times the instanton density, so a $U(1)_R$ rotation by α shifts the theta angle by $2N_c\,\alpha$. Rotations that shift θ by a multiple of 2π are unbroken, which leaves

$$U(1)_R \;\longrightarrow\; \mathbb{Z}_{2N_c}. \tag{I.3.20}$$

This $\mathbb{Z}_{2N_c}$ is an exact symmetry of the quantum theory.

Gaugino condensation and the N_c vacua. The strong dynamics forms a gaugino condensate, a nonzero vacuum expectation value of the lowest component of the gauge-kinetic chiral operator,

$$\langle\, \lambda\lambda \,\rangle_k \;\sim\; \Lambda^3\, e^{\,2\pi i k/N_c}, \qquad k = 0, 1, \ldots, N_c - 1. \tag{I.3.21}$$

The operator $\lambda\lambda$ has R-charge two, so a $\mathbb{Z}_{2N_c}$ generator acts on it by

$$\lambda\lambda \;\longmapsto\; e^{\,2\pi i\cdot 2/(2N_c)}\,\lambda\lambda \;=\; e^{\,2\pi i/N_c}\,\lambda\lambda, \tag{I.3.22}$$

which fixes the condensate only for the subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_{2N_c}$ (the elements that fix a charge-two operator), so $\mathbb{Z}_{2N_c}$ is spontaneously broken down to $\mathbb{Z}_2$. The broken discrete symmetry relates

$$\frac{|\mathbb{Z}_{2N_c}|}{|\mathbb{Z}_2|} \;=\; \frac{2N_c}{2} \;=\; N_c \tag{I.3.23}$$

inequivalent vacua, the N_c distinct roots of Λ^{3N_c} displayed in (I.3.21).

None of these vacua breaks supersymmetry. The condensate is *not* a supersymmetry-breaking order parameter: $\lambda\lambda$ is the lowest component of a chiral operator, its vacuum expectation value is consistent with a zero-energy state, and supersymmetry is unbroken in each of the N_c vacua. The condensate labels the distinct vacua produced by the discrete R-symmetry breaking; it does not indicate broken supersymmetry. (Phrasing this as "an F-flat configuration that breaks an R-symmetry" is the careful statement; calling the condensate itself a supersymmetry-breaking order parameter is the trap.)

The Witten index. The count of supersymmetric vacua is the Witten index,

$$\mathrm{Tr}\,(-1)^F \;=\; N_c \;\neq\; 0, \tag{I.3.24}$$

consistent with the $|\mathbb{Z}_{2N_c}|/|\mathbb{Z}_2| = N_c$ condensate vacua. The index is a deformation invariant, so the same number must emerge from a completely different (and weakly coupled) computation. Compactify the theory on a spatial three-torus of small volume. The gauge field then has flat connections labelled by commuting holonomies around the three cycles, and for $SU(N_c)$ quantization of the flat-connection zero modes gives index N_c (the dual Coxeter number, which for $SU(N_c)$ equals N_c). The weak-coupling count and the strong-coupling gaugino-condensate count agree, as the deformation invariance of the index demands. A nonzero Witten index forbids dynamical supersymmetry breaking, so pure super Yang–Mills confines with N_c gaugino-condensate vacua and unbroken supersymmetry.

Optional: the glueball superpotential. The same N_c vacua follow from the Veneziano–Yankielowicz effective superpotential for the glueball chiral superfield $S = -\frac{1}{32\pi^2}\,\mathrm{Tr}\,W_\alpha W^\alpha$ (whose lowest component is $\sim \lambda\lambda$),

$$W_{\rm VY}(S) \;=\; S\left[\,N_c - \ln\frac{S^{N_c}}{\Lambda^{3N_c}}\,\right]. \tag{I.3.25}$$

Its F-term equation locates the vacua,

$$\frac{\partial W_{\rm VY}}{\partial S} = -\ln\frac{S^{N_c}}{\Lambda^{3N_c}} = 0 \implies S^{N_c} = \Lambda^{3N_c} \implies \langle S\rangle = \Lambda^3\, e^{2\pi i k/N_c}, \tag{I.3.26}$$

the N_c roots recovering (I.3.21). Substituting back,

$$W_{\rm VY}(\langle S\rangle) = N_c\langle S\rangle = N_c\,\Lambda^3\, e^{2\pi i k/N_c}, \tag{I.3.27}$$

so the different values of the effective superpotential distinguish the N_c vacua and agree with the discrete-symmetry count. We use it only to locate the vacua; the full glueball effective action and its derivation are developed in Part III.

Tying the SYM scale to SQCD. The pure-SYM scale is not an independent input: it is the result of successively removing the SQCD flavors. Iterating (I.3.9) from N_f flavors down to zero, giving each flavor a mass m_a,

$$\Lambda_{\rm SYM}^{3N_c} \;=\; \Big(\textstyle\prod_{a=1}^{N_f} m_a\Big)\,\Lambda_{N_f}^{3N_c-N_f}, \tag{I.3.28}$$

the exponent on the left being $b_0|_{N_f=0} = 3N_c$. The gaugino condensate of the resulting pure SYM is proportional to the N_c-th root of (I.3.28),

$$\langle\lambda\lambda\rangle \;\sim\; \Lambda_{\rm SYM}^3 \;=\; \Big(\textstyle\prod_a m_a\Big)^{1/N_c}\,\Lambda_{N_f}^{(3N_c-N_f)/N_c}, \tag{I.3.29}$$

expressed entirely in the SQCD data and the masses. This is the $\mathcal{N}=1$ holomorphic-scale relation that the next section's R-symmetry and the duality of §I.3.7 both rely on, and it is the construction the scale-matching pointer below contrasts with the $\mathcal{N}=2$ case.

A scale-matching pointer, stated carefully. It is tempting to read the $\mathcal{N}=2$ Seiberg–Witten scale of the $4d$ $\mathcal{N}=2$ chapter as this same $\mathcal{N}=1$ condensate. It is not. The $\mathcal{N}=2$ scale is generated by $\mathcal{N}=2$ dimensional transmutation, the running of the $\mathcal{N}=2$ gauge theory, with its own one-loop coefficient. The two scales are related only after an $\mathcal{N}=1$ mass deformation: giving the $\mathcal{N}=2$ adjoint chiral a mass and integrating it out (a threshold matching of the type (I.3.9), but across the $\mathcal{N}=2 \to \mathcal{N}=1$ decoupling) leaves pure $\mathcal{N}=1$ super Yang–Mills, and the holomorphic matching of scales across that decoupling ties the two together. The $4d$ $\mathcal{N}=2$ chapter develops the $\mathcal{N}=2$ side. Here we only flag the seam: do not equate the $\mathcal{N}=2$ transmutation scale with the $\mathcal{N}=1$ condensate before the mass deformation is performed.

I.3.4 R-symmetry, NSVZ, and the superconformal R

A $U(1)_R$ R-symmetry is part of the superconformal algebra at a fixed point, as Chapter 1 showed. The relevant convention is the fermion R-shift of Chapter 2. The gauge-kinetic superfield has R-charge two, so the gaugino fermion has $R = 1$. A chiral multiplet whose scalar carries R-charge R_s has its fermion at

$$R_\psi \;=\; R_s - 1. \tag{I.3.30}$$

Anomaly traces are fermion traces, and forgetting this shift is the most common way to get the wrong answer. We will use it on every trace in this chapter.

The NSVZ beta function. The exact running of the physical (canonically normalized) coupling is the NSVZ beta function,

$$\beta(g) \;=\; -\frac{g^3}{16\pi^2}\,\frac{3\,T(\text{adj}) \;-\; \sum_i T(r_i)\,(1-\gamma_i)}{1 \;-\; \dfrac{g^2\,T(\text{adj})}{8\pi^2}}, \tag{I.3.31}$$

with γ_i the anomalous dimension of the chiral field in representation r_i. The numerator generalizes b_0 of (I.3.2) by the matter anomalous dimensions; the denominator is the all-orders correction from the holomorphic-versus-physical rescaling. Its origin is one computation: the holomorphic coupling runs only at one loop, but passing to the canonically normalized coupling requires a field rescaling whose Jacobian is anomalous (a Konishi-type anomaly), and that anomaly is exactly the geometric series $1/(1 - g^2T(\text{adj})/8\pi^2)$. The one-loop running plus the rescaling anomaly is the full physical beta function, with no independent higher-loop input. At a fixed point the beta function vanishes, and since the denominator is finite the numerator vanishes:

$$3\,T(\text{adj}) \;=\; \sum_i T(r_i)\,(1-\gamma_i). \tag{I.3.32}$$

This fixes the γ_i at the fixed point, and the fixed-point R-charge of a chiral field follows from the superconformal relation

$$R_i \;=\; \frac{2}{3}\Big(1 + \tfrac{1}{2}\gamma_i\Big), \tag{I.3.33}$$

which encodes a chiral primary of dimension

$$\Delta_i \;=\; 1 + \tfrac{1}{2}\gamma_i \;=\; \tfrac{3}{2}R_i \quad \text{at the fixed point.} \tag{I.3.34}$$

A free field has $\gamma_i = 0$ and $R_i = \frac{2}{3}$; an interacting fixed point shifts γ_i away from zero and R_i away from $\frac{2}{3}$, and (I.3.33) is the dictionary between the two. Reading (I.3.32) and (I.3.33) together: the NSVZ condition is one equation in the anomalous dimensions, the R-dictionary turns it into one equation in the R-charges, and for SQCD it is the same condition as gauge-R anomaly freedom.

The SQCD quark R-charge, derived. For $SU(N_c)$ SQCD with N_f flavors the flavor symmetry forces all the quarks to share one R-charge, $R(Q) = R(\widetilde{Q}) \equiv R_Q$.

The NSVZ condition (I.3.32) is then equivalent to the freedom of the mixed gauge-R (ABJ) anomaly, $SU(N_c)^2\,U(1)_R = 0$. That anomaly receives the gaugino contribution $T(\text{adj})$ at $R_\psi = 1$ and, from each of the $2N_f$ fundamental quark fermions, $T(\square)$ at $R_\psi = R_Q - 1$:

$$T(\text{adj}) \;+\; N_f\,T(\square)\,(R_Q - 1) \;+\; N_f\,T(\square)\,(R_{\widetilde{Q}} - 1) \;=\; 0. \tag{I.3.35}$$

With $T(\text{adj}) = N_c$, $T(\square) = \frac{1}{2}$, and $R_Q = R_{\widetilde{Q}}$, this collapses to

$$N_c \;+\; N_f\,(R_Q - 1) \;=\; 0, \tag{I.3.36}$$

so

$$R(Q) \;=\; 1 - \frac{N_c}{N_f}, \tag{I.3.37}$$

the symmetry-determined superconformal R-charge of the SQCD quark, and the quark *fermion* carries $R_\psi = R(Q) - 1 = -N_c/N_f$. This is the data the 't Hooft anomalies of §I.3.5 and the central charges of §I.3.6 are built from.

The naive guess $R(Q) = \frac{1}{2}$ does *not* solve (I.3.35) except at the special point $N_f = 2N_c$,

$$R(Q)\big|_{N_f=2N_c} = 1 - \tfrac{N_c}{2N_c} = \tfrac{1}{2}, \qquad R(Q)\big|_{N_c=3,\,N_f=5} = 1 - \tfrac{3}{5} = \tfrac{2}{5} \neq \tfrac{1}{2}. \tag{I.3.38}$$

And $R(Q) = 0$ (a "free massless quark, no R-charge") leaves the gaugino's $+N_c$ uncancelled in (I.3.36). The superconformal R is forced; it is not whatever value looks symmetric.

What the superconformal R is. The superconformal R is a specific object, not a free choice: it is the one linear combination of the classical R-symmetry and the abelian flavor symmetries that sits inside the superconformal multiplet. The naive ultraviolet assignment ($R = \frac{2}{3}$ or equal R-charges) is generally *not* it unless a symmetry forces it. On a chiral primary the dimension is locked to this R by $\Delta = \frac{3}{2}R$, the saturated unitarity bound of Chapter 1, computed with the superconformal R, never the ultraviolet one. Symmetry does the work when the candidate R is unique up to constraints, as in SQCD (one flavor symmetry and one anomaly condition) or a quiver (a discrete symmetry forcing a symmetric assignment). When the candidate R has a free parameter, as in adjoint SQCD, the selection is the extremization of §I.3.6, and the answer is generally not a naive guess.

I.3.5 't Hooft anomalies and the Konishi anomaly

The 't Hooft anomalies of the continuous global symmetries are renormalization-group invariants. They are computed from the ultraviolet fermion content, they cannot change along the flow, and they must equal the value computed in any infrared description. Matching them between a proposed ultraviolet and infrared (or electric and magnetic) frame is therefore a necessary consistency test. It is necessary, not sufficient: two theories can share every 't Hooft anomaly and still be different, so anomaly matching

is *evidence* for an identification, never a proof of it. This is the gauge-versus-'t Hooft distinction of Chapter 2 turned into a working tool.

The SQCD global symmetry and its conventions. For $SU(N_c)$ SQCD with N_f flavors the continuous global symmetry is

$$SU(N_f)_L \;\times\; SU(N_f)_R \;\times\; U(1)_B \;\times\; U(1)_R. \tag{I.3.39}$$

We declare the representation convention once and keep it for the whole chapter. The quarks Q are a fundamental of $SU(N_f)_L$ and a singlet of $SU(N_f)_R$; the antiquarks $\widetilde{Q}$ are an antifundamental of $SU(N_f)_R$ and a singlet of $SU(N_f)_L$. Under $U(1)_B$, Q has charge $+1$ and $\widetilde{Q}$ has charge -1. The R-charges are $R(Q) = R(\widetilde{Q}) = 1 - N_c/N_f$, so the quark fermions carry $R_\psi = -N_c/N_f$ ((I.3.37), (I.3.30)); the gaugino is at $R_\psi = 1$. The full assignment is collected in Table I.3.1.

field	$SU(N_c)$ gauge	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	$\mathbf{1}$	$+1$	$1 - N_c/N_f$
$\widetilde{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$1 - N_c/N_f$
λ (gaugino)	adj	$\mathbf{1}$	$\mathbf{1}$	0	$+1$

Table I.3.1: The electric $SU(N_c)$ SQCD matter content and global charges. The quark and antiquark fermions carry $R_\psi = R(Q) - 1 = -N_c/N_f$; the gaugino fermion carries $R_\psi = +1$. The $SU(N_f)_R$ acts on $\widetilde{Q}$ as an antifundamental, which fixes the sign of its cubic anomaly.

The cubic flavor anomalies, with the L/R split. Only the N_c colors of a quark are charged under a flavor group, so a cubic flavor anomaly counts colors. Because Q is a fundamental of $SU(N_f)_L$ and $\widetilde{Q}$ is an *antifundamental* of $SU(N_f)_R$, the two cubic anomalies have opposite signs:

$$SU(N_f)_L^3 \;=\; +N_c, \qquad SU(N_f)_R^3 \;=\; -N_c, \tag{I.3.40}$$

each from the N_c colors of the corresponding quark multiplied by the cubic index of a (anti)fundamental (± 1). (Had one conjugated $\widetilde{Q}$ into a fundamental of $SU(N_f)_R$ instead, both signs would be $+N_c$; we use the antifundamental convention of Table I.3.1 throughout.) Splitting the two flavor groups is not cosmetic: the sign is exactly what the Seiberg dual must reproduce, and a single "$SU(N_f)^3 = N_c$" obscures which group is which.

The R-traces, run explicitly. The gravitational and cubic R-anomalies are pure fermion traces over R_ψ and R_ψ^3. The gravitational trace $\mathrm{Tr}\, R$ sums the $N_c^2 - 1$ gaugino components at $R_\psi = 1$ and the $2N_fN_c$ quark fermions at $R_\psi = -N_c/N_f$:

$$\mathrm{Tr}\, R \;=\; (N_c^2 - 1)\cdot 1 \;+\; 2N_fN_c\left(-\frac{N_c}{N_f}\right) \;=\; (N_c^2 - 1) - 2N_c^2 \;=\; -(N_c^2 + 1). \tag{I.3.41}$$

The cubic trace $\mathrm{Tr}\, R^3$ uses the same fermions at the cube of their R-charge:

$$\mathrm{Tr}\, R^3 \;=\; (N_c^2-1)\cdot 1^3 \;+\; 2N_fN_c\left(-\frac{N_c}{N_f}\right)^3 \;=\; (N_c^2-1)-\frac{2N_c^4}{N_f^2}. \tag{I.3.42}$$

The mixed flavor-R trace counts only the quark charged under the flavor group, with the fundamental index $T(\square)=\frac{1}{2}$:

$$SU(N_f)_L^2\, U(1)_R \;=\; N_c\cdot\tfrac{1}{2}\cdot\left(-\frac{N_c}{N_f}\right) \;=\; -\frac{N_c^2}{2N_f}, \tag{I.3.43}$$

and the same value (with the antifundamental of $\widetilde{Q}$) for $SU(N_f)_R^2\, U(1)_R$. Two more mixed traces complete the list the duality will test. The flavor-baryon trace counts the colors with the baryon charge ± 1,

$$SU(N_f)_L^2\, U(1)_B \;=\; N_c\cdot\tfrac{1}{2}\cdot(+1) \;=\; \frac{N_c}{2}, \tag{I.3.44}$$

and the cubic-baryon-R trace is

$$U(1)_B^2\, U(1)_R \;=\; 2N_fN_c\cdot(+1)^2\cdot\left(-\frac{N_c}{N_f}\right) \;=\; -2N_c^2, \tag{I.3.45}$$

summing both quark species (the gaugino is $U(1)_B$-neutral). These are the anomalies the Seiberg dual of §I.3.7 must reproduce trace by trace, and the central charges of §I.3.6 are built from (I.3.41) and (I.3.42) directly.

Worked numbers. At $N_c=3, N_f=6$ the quark fermion R-charge is $-N_c/N_f=-\frac{1}{2}$, so

$$\begin{aligned}\mathrm{Tr}\, R\big|_{3,6} &= (9-1)-2\cdot 6\cdot 3\cdot\tfrac{1}{2} = 8-18=-10=-(N_c^2+1),\\ \mathrm{Tr}\, R^3\big|_{3,6} &= 8-36\cdot\tfrac{1}{8} = 8-\tfrac{9}{2}=\tfrac{7}{2},\end{aligned} \tag{I.3.46}$$

confirming (I.3.41) and (I.3.42). These two numbers feed directly into the central charges of the next section.

The Konishi anomaly. The Konishi anomaly is the operator equation that ties the matter sector to the gauge field strength. Classically the Konishi current $\bar{\Phi}e^V\Phi$ is conserved up to the superpotential; quantum mechanically it acquires a gauge-anomaly divergence,

$$\bar{D}^2\big(\bar{\Phi}e^V\Phi\big) \;=\; \Phi\,\frac{\partial W}{\partial\Phi} \;+\; \frac{T(r)}{8\pi^2}\,\mathrm{Tr}\, W_\alpha W^\alpha. \tag{I.3.47}$$

We normalize $\bar{D}^2$ so that the classical piece reads $\Phi\,\partial W/\partial\Phi$; its coefficient is convention-dependent, while the load-bearing *anomaly* coefficient $T(r)/8\pi^2$ is not. That anomaly coefficient multiplying $\mathrm{Tr}\, W_\alpha W^\alpha$ is the *same* gauge Dynkin index $T(r)$ that enters the mixed gauge-R (ABJ) anomaly fixing $R(Q)$ above. That shared index is what makes the Konishi equation consistent with the R-charge calculation: the per-flavor matter contribution to the gauge-R anomaly is $T(\square)$ times the quark fermion R-charge, with

$T(\square) = \frac{1}{2}$, not 1. To see the consistency as arithmetic, sum the Konishi equation over the $2N_f$ fundamentals. The matter contribution to the divergence of the R-current, plus the gaugino's $T(\text{adj}) = +N_c$, gives

$$\underbrace{2N_f\, T(\square)\, R_\psi(Q)}_{\text{matter}} + \underbrace{T(\text{adj})}_{\text{gaugino}} \;=\; 2N_f \cdot \tfrac{1}{2} \cdot \Big(-\tfrac{N_c}{N_f}\Big) + N_c \;=\; -N_c + N_c \;=\; 0, \qquad \text{(I.3.48)}$$

exactly (I.3.35). The Konishi and gauge-R Ward identities are distinct. Their gauge-anomaly terms involve the same representation index $T(\square) = \frac{1}{2}$, which makes the displayed coefficients consistent. The Konishi anomaly feeds the matter anomalous dimensions and the chiral-ring relations. We state it as the operator equation and give its derivation in Part III.

Common misconception: anomaly matching is necessary, not a proof

Equal 't Hooft anomalies are a *necessary* condition for two theories to flow to the same infrared physics, never a sufficient one. Two distinct theories can carry identical anomaly polynomials. When a proposed duality passes the anomaly match, that is real evidence, weighed together with the chiral-ring and flavor-symmetry matches, but it does not by itself prove the duality. The proof, where one exists, is a separate argument taken up in Part III.

I.3.6 a-maximization: the extremization prototype

The two conformal central charges of a $4d$ $\mathcal{N} = 1$ fixed point are built from the same fermion traces over a trial R-symmetry,

$$a(R) = \frac{3}{32}\big(3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R\big), \qquad c(R) = \frac{1}{32}\big(9\,\mathrm{Tr}\,R^3 - 5\,\mathrm{Tr}\,R\big), \qquad \text{(I.3.49)}$$

with the gaugino at $R = 1$ and a chiral fermion at $R_s - 1$, the prescription of Chapter 2. The exact superconformal R is the trial R that *locally maximizes* a. It is a that is extremized, not c, and the extremum is a maximum, not a minimum. This is the prototype of the extremization through-line of Volume 0: the $2d$ chapter does c-extremization as the two-dimensional analog of this section, and the $3d$ chapter does F-maximization as the three-dimensional analog. Both point back here.

These central charges are not arbitrary combinations: a and c are the coefficients of the Euler and Weyl-squared terms in the trace anomaly of the stress tensor on a curved background, and supersymmetry ties them to the 't Hooft anomalies of the R-current through (I.3.49). A single free chiral multiplet has its fermion at $R = -\frac{1}{3}$, and a free vector multiplet has its gaugino at $R = 1$, giving the standard free-field values

$$\begin{aligned} &\text{chiral:} \quad \mathrm{Tr}\,R = -\tfrac{1}{3}, \;\; \mathrm{Tr}\,R^3 = -\tfrac{1}{27} \implies a = \tfrac{1}{48},\; c = \tfrac{1}{24}, \\ &\text{vector:} \quad \mathrm{Tr}\,R = 1, \;\; \mathrm{Tr}\,R^3 = 1 \implies a = \tfrac{3}{16},\; c = \tfrac{1}{8}. \end{aligned} \qquad \text{(I.3.50)}$$

These free-field values check the sign and normalization of any trace. They also exhibit $a \neq c$ already at free-field level, so the inequality is the rule, not a pathology.

The maximization is constrained, not free. A trial R ranges over the abelian symmetries that respect gauge-anomaly freedom (the mixed gauge-R, or ABJ, condition $SU(N_c)^2$-$U(1)_R = 0$) and that assign R-charge two to every superpotential term, together with any flavor symmetry one chooses to impose. Parametrize the trial R as $R_t = R_0 + \sum_I s_I F_I$, with R_0 a reference R-charge, F_I the abelian flavor charges, and s_I the free mixing parameters; the constrained extremization is

$$\frac{\partial a}{\partial s_I} \;=\; 0, \qquad \frac{\partial^2 a}{\partial s_I\, \partial s_J} \;<\; 0 \text{ (negative definite)}, \tag{I.3.51}$$

a stationarity equation in the mixing parameters with the Hessian condition selecting the maximum. The cubic and gravitational 't Hooft anomalies $\mathrm{Tr}\, R^3$ and $\mathrm{Tr}\, R$ are the *data* that build a and c; they are not constraints that must vanish. One further condition matters. An operator whose dimension would fall below the unitarity bound decouples as a free field, and the maximization must be redone without it. The accidental-symmetry correction is developed in Part III.

Worked instance: SQCD

Take $SU(N_c)$ SQCD in the conformal window. Here symmetry already fixes the trial R: one quark R-charge, one anomaly condition, giving $R(Q) = 1 - N_c/N_f$ as in (I.3.37), so the quark fermion carries $R_\psi = -N_c/N_f$. The two traces are already computed, (I.3.41) and (I.3.42),

$$\mathrm{Tr}\, R = -(N_c^2+1), \qquad \mathrm{Tr}\, R^3 = (N_c^2-1) - \frac{2N_c^4}{N_f^2}, \tag{I.3.52}$$

and (I.3.49) then gives a and c as explicit rational functions of (N_c, N_f). Substituting (I.3.52),

$$\begin{aligned} a(N_c,N_f) &= \frac{3}{32}\left[3(N_c^2-1) - \frac{6N_c^4}{N_f^2} + (N_c^2+1)\right] = \frac{3}{32}\left[4N_c^2 - 2 - \frac{6N_c^4}{N_f^2}\right], \\ c(N_c,N_f) &= \frac{1}{32}\left[9(N_c^2-1) - \frac{18N_c^4}{N_f^2} + 5(N_c^2+1)\right] = \frac{1}{32}\left[14N_c^2 - 4 - \frac{18N_c^4}{N_f^2}\right]. \end{aligned} \tag{I.3.53}$$

These are distinct rational functions: $a \neq c$ in general. Subtracting the two formulas of (I.3.49) gives

$$a - c \;=\; \tfrac{1}{16}\,\mathrm{Tr}\, R, \tag{I.3.54}$$

so the central charges coincide precisely when $\mathrm{Tr}\, R = 0$ (for example $\mathcal{N} = 4$ super Yang–Mills), a special property, not a generic feature. For the explicit case $N_c = 3, N_f = 6$, with $\mathrm{Tr}\, R = -10$ and $\mathrm{Tr}\, R^3 = \frac{7}{2}$ from (I.3.46),

$$\begin{aligned} a(3,6) &= \frac{3}{32}\big(3\cdot\tfrac{7}{2} + 10\big) = \frac{3}{32}\cdot\frac{41}{2} = \frac{123}{64}, \\ c(3,6) &= \frac{1}{32}\big(9\cdot\tfrac{7}{2} + 50\big) = \frac{1}{32}\cdot\frac{163}{2} = \frac{163}{64}. \end{aligned} \tag{I.3.55}$$

Both are finite, positive, and unitary, and $\frac{123}{64} \neq \frac{163}{64}$ exhibits $a \neq c$ on a concrete case. Running the same arithmetic across the $N_c = 3$ conformal window ($\frac{9}{2} < N_f < 9$) gives Table I.3.2: $a < c$ in every entry (consistent with $\frac{1}{2} \leq a/c \leq \frac{3}{2}$), and a/c drifts from 0.686 at the bottom toward 0.798 at the top, approaching the free-field ratio as the dynamics weakens. Each number is (I.3.53) evaluated at the listed (N_c, N_f), computed not quoted.

N_f	$\mathrm{Tr}\, R$	$\mathrm{Tr}\, R^3$	a	c	a/c
5	-10	38/25	$273/200 \approx 1.365$	$199/100 \approx 1.990$	0.686
6	-10	7/2	$123/64 \approx 1.922$	$163/64 \approx 2.547$	0.755
7	-10	230/49	$885/392 \approx 2.258$	$565/196 \approx 2.883$	0.783
8	-10	175/32	$2535/1024 \approx 2.476$	$3175/1024 \approx 3.101$	0.798

Table I.3.2: The SQCD central charges a, c across the $N_c = 3$ conformal window, computed from (I.3.53). The gravitational trace $\mathrm{Tr}\, R = -(N_c^2 + 1) = -10$ is N_f-independent; $\mathrm{Tr}\, R^3 = 8 - 2 \cdot 81/N_f^2$ carries the N_f dependence. In every entry $a < c$ and a/c rises toward the free-field value as the window edge is approached.

Because the trial R has no free parameter here, there is nothing left to maximize: the symmetric assignment already is the superconformal R. SQCD shows the machinery on a case where the answer is rational and symmetry-forced.

Worked instance: adjoint SQCD, the genuine extremization

Now add one adjoint chiral field X to $SU(N_c)$ with N_f fundamentals (with no superpotential, so R_X is free; turning on $W = \mathrm{Tr}\, X^{k+1}$ instead fixes $R_X = 2/(k+1)$ by the superpotential constraint and removes the free parameter, a different problem). The trial R now has a free parameter: the adjoint R-charge R_X is not fixed by symmetry alone. Only the ABJ condition relates the quark R-charge to it. The mixed gauge-R anomaly now receives the gaugino, the adjoint chiral X (also index $T(\mathrm{adj}) = N_c$, at $R_\psi = R_X - 1$), and the $2N_f$ fundamentals:

$$N_c \;+\; N_c\,(R_X - 1) \;+\; N_f\,(R_Q - 1) \;=\; 0 \quad\Longrightarrow\quad R_Q \;=\; 1 - \frac{N_c}{N_f}\,R_X. \tag{I.3.56}$$

With R_Q slaved to R_X, one builds the constrained $a(R_X)$ from the fermion traces. Write $r_X \equiv R_X - 1$ for the adjoint fermion R-charge and $r_Q \equiv R_Q - 1 = -\frac{N_c}{N_f} R_X$ for the quark fermion R-charge, from (I.3.56). The gravitational trace is

$$\mathrm{Tr}\, R \;=\; \underbrace{(N_c^2 - 1)}_{\text{gaugino}} \;+\; \underbrace{(N_c^2 - 1)\, r_X}_{\text{adjoint } X} \;+\; \underbrace{2N_f N_c\, r_Q}_{\text{quarks}}, \tag{I.3.57}$$

and the cubic trace uses the cubes of the same fermion R-charges,

$$\mathrm{Tr}\, R^3 \;=\; (N_c^2 - 1) \;+\; (N_c^2 - 1)\, r_X^3 \;+\; 2N_f N_c\, r_Q^3. \tag{I.3.58}$$

Because $r_Q = -\frac{N_c}{N_f} R_X$ is linear in R_X and $r_X = R_X - 1$ is linear in R_X, the trace $\mathrm{Tr}\, R^3$ is a *cubic* polynomial in R_X, and so is the trial central charge

$$a(R_X) \;=\; \frac{3}{32}\big(3\,\mathrm{Tr}\, R^3 - \mathrm{Tr}\, R\big). \tag{I.3.59}$$

The stationarity equation

$$\frac{da}{dR_X} \;=\; 0 \tag{I.3.60}$$

is therefore a *quadratic* in R_X. The trial a is cubic, but its derivative is quadratic. Thus the stationary R-charge solves a quadratic equation, not a cubic one. (Calling it "a root of a cubic" double-counts a degree; the cubic object is a itself, the equation to solve is quadratic.) A quadratic with real coefficients has two real roots when its discriminant is positive; the physical superconformal R is the one with $d^2a/dR_X^2 < 0$ (a local maximum) that respects the unitarity bounds. A quadratic root can be irrational when its discriminant is not a rational square. The example below has this property. Run it at $N_c = 3, N_f = 4$. The gauge-anomaly constraint gives $r_Q = -\frac{3}{4} R_X$, and assembling (I.3.57) and (I.3.58) the trial central charge is

$$a(R_X) \;=\; -\frac{153}{256}\, R_X^3 \;-\; \frac{27}{4}\, R_X^2 \;+\; \frac{123}{16}\, R_X, \tag{I.3.61}$$

a genuine cubic. Its derivative is the quadratic

$$\frac{da}{dR_X} \;=\; -\frac{459}{256}\, R_X^2 \;-\; \frac{27}{2}\, R_X \;+\; \frac{123}{16} \;=\; 0, \tag{I.3.62}$$

whose two roots are $R_X = -\frac{64}{17} \pm \frac{4}{51}\sqrt{3001}$. The physical one is

$$R_X^\star \;=\; -\frac{64}{17} + \frac{4}{51}\sqrt{3001} \;\approx\; 0.5319, \tag{I.3.63}$$

manifestly irrational (it carries the surd $\sqrt{3001}$), in contrast to the symmetry-forced SQCD value. The other root, $R_X \approx -8.06$, is unphysical. The second derivative

$$a''(R_X^\star) \;=\; -\tfrac{459}{128}\, R_X^\star \;-\; \tfrac{27}{2} \;<\; 0 \tag{I.3.64}$$

confirms it is a maximum, and perturbing R_X either way lowers a. The same R_X does *not* extremize $c(R_X) = \frac{1}{32}(9\,\mathrm{Tr}\, R^3 - 5\,\mathrm{Tr}\, R)$ (also cubic in R_X): $dc/dR_X|_{R_X^\star} \neq 0$, the c-maximizing root sits at a different value, and the value of c at the a-maximum lies strictly below the c-maximum. This is the sharpest illustration that one maximizes a and not c. It is the case where the extremization earns its name: the answer is selected by the maximum of a, not read off a symmetry.

Worked instance: the dP_0 quiver

The $\mathbb{C}^3/\mathbb{Z}_3$ (local $\mathbb{P}^2$, or dP_0) daughter that Volume V builds is a three-node $\prod_i U(N_i)$ quiver with nine chiral bifundamentals $X^a_{i,i+1}$ $(a = 1, 2, 3)$ and the cubic superpotential

$W = \epsilon_{abc}\,\mathrm{Tr}\,X_{01}^a X_{12}^b X_{20}^c$. Its R-symmetry is forced two independent ways. The superpotential carries R-charge two, and a cubic word in the bifundamentals carries R-charge $3R(X)$, so

$$R(W) = 3\,R(X) = 2 \quad \Longrightarrow \quad R(X) = \tfrac{2}{3}. \tag{I.3.65}$$

Independently, the per-node mixed gauge-R anomaly cancels at the symmetric assignment. At node i the fields charged under that gauge factor are its gaugino (index N at $R_\psi = 1$) and the six bifundamentals touching it (three incoming, three outgoing, each contributing $N \cdot \frac{1}{2}$ at $R_\psi = R(X) - 1$):

$$N \;+\; 6 \cdot N \cdot \tfrac{1}{2}\,(R(X) - 1) \;=\; N\big[\,1 + 3(R(X) - 1)\,\big] \;=\; 0 \quad \Longrightarrow \quad R(X) = \tfrac{2}{3}, \tag{I.3.66}$$

independent of N. In the symmetric sector the superpotential constraint already fixes $R(X) = \frac{2}{3}$; there is no remaining common-R direction along which to maximize. The anomaly condition checks the same assignment independently. A variation that changes all nine charges together violates $R(W) = 2$, so its Hessian is not a physical a-maximization test. With the R-charge fixed, the central charge is a fermion trace over the three nodes. Take $N \geq 2$ and the gauge group $\prod_{i=0}^{2} SU(N)$ (the interacting factor, the overall diagonal $U(1)$ decoupled). Each node contributes a gaugino at $R = 1$ (dimension $N^2 - 1$), and each of the nine bifundamentals contributes a fermion at $R_\psi = R(X) - 1 = -\frac{1}{3}$ (dimension N^2):

$$\mathrm{Tr}\,R = 3(N^2-1)\cdot 1 + 9N^2\big(-\tfrac{1}{3}\big) = -3, \qquad \mathrm{Tr}\,R^3 = 3(N^2-1) + 9N^2\big(-\tfrac{1}{3}\big)^3 = \tfrac{8}{3}N^2 - 3, \tag{I.3.67}$$

and assembling (I.3.49) the central charges of the dP_0 quiver as functions of the rank are

$$a \;=\; \tfrac{3}{32}\big(3(\tfrac{8}{3}N^2 - 3) + 3\big) \;=\; \tfrac{3}{4}N^2 - \tfrac{9}{16}, \qquad c \;=\; \tfrac{3}{4}N^2 - \tfrac{3}{8}, \tag{I.3.68}$$

with $a \neq c$. (At large N both grow as $\frac{3}{4}N^2$, the leading planar inheritance from the $\mathcal{N} = 4$ parent; the subleading constants distinguish the daughter.) The numbers displayed here are the *interacting* central charges with the decoupled diagonal $U(1)$ removed; one must state which it is, because including the free diagonal $U(1)$ shifts a by the free-vector value $\frac{3}{16}$. This example joins the field-theory calculation to the brane construction. Volume V constructs the $\prod_i U(N_i)$ quiver from a D-brane probe. Its symmetric R-charge and the interacting-sector central charges then follow from the displayed field content. The orbifold projection supplies that content; the anomaly calculation begins with it.

Worked entry point: the Banks–Zaks fixed point

The cleanest place to see that the conformal window contains genuine fixed points is its upper edge, where the dynamics is weakly coupled and the fixed point is perturbatively accessible. Take large N_c and N_f just below the asymptotic-freedom bound,

$$N_f \;=\; 3N_c - \epsilon\,N_c, \qquad 0 < \epsilon \ll 1, \tag{I.3.69}$$

so $b_0 = 3N_c - N_f = \epsilon N_c$ is positive but small. The two-loop beta function

$$\beta \;=\; -\frac{g^3}{16\pi^2}\,b_0 \;-\; \frac{g^5}{(16\pi^2)^2}\,b_1 \;+\; \cdots, \tag{I.3.70}$$

with the two-loop coefficient b_1 negative in this regime, has its two contributions cancel at

$$g_*^2 \;=\; -\frac{16\pi^2\, b_0}{b_1} \;\sim\; \frac{\epsilon}{N_c}, \tag{I.3.71}$$

a zero of β that is positive and parametrically small. This is the Banks–Zaks fixed point: a weakly coupled, controllably computable $4d$ $\mathcal{N}=1$ superconformal field theory at the top of the window. The fixed point is found by solving the two-loop beta-function zero, not asserted; at N_f outside the window the would-be zero (I.3.71) sits at strong or negative coupling and is not a controlled fixed point. Because the coupling is small, perturbation theory controls the anomalous dimensions and correlation functions. The exact infrared R-charge is still the anomaly/a-maximization value

$$R(Q) \;=\; 1-\frac{N_c}{N_f} \;=\; \frac{2}{3}-\frac{\epsilon}{9}+O(\epsilon^2), \tag{I.3.72}$$

so it approaches the free value $R=\frac{2}{3}$ as $\epsilon\to 0$; the weak-coupling expansion is the controlled realization of that exact R-symmetry, not a separate definition of it. The Banks–Zaks point is therefore the controlled entry point to the conformal window: it is where a-maximization, the NSVZ condition, and the central charges are all computable in a small parameter, and it anchors the rest of the window by continuity.

Common misconception: a-maximization maximizes a, with constraints

The extremized quantity is a, not c, and the extremum is a local *maximum*, not a minimum or an unconstrained extremum. The maximization runs over trial R-symmetries constrained by gauge-anomaly freedom (the ABJ condition $SU(N_c)^2$-$U(1)_R=0$) and by every superpotential term carrying R-charge two; the cubic and gravitational 't Hooft anomalies are the *data* that build the central charges, not constraints that must vanish. The superconformal R is generally not the naive ultraviolet assignment: it equals it only when symmetry leaves no free parameter (SQCD, the dP_0 quiver), and is selected by the maximum otherwise (adjoint SQCD, where it is irrational). Operators that hit the unitarity bound must be decoupled and the maximization redone.

The window edges and meson unitarity

The two edges of the conformal window are not arbitrary. They are fixed by the unitarity bound on chiral operators that Chapter 1 established: a gauge-invariant chiral primary obeys $\Delta\geq 1$, with $\Delta=1$ exactly for a free field, and a chiral primary saturates $\Delta=\frac{3}{2}R$. At the interacting fixed point the quark superfield carries the anomaly-fixed R-charge

$$R(Q) \;=\; 1-\frac{N_c}{N_f}, \tag{I.3.73}$$

so the meson $M=Q\widetilde{Q}$, a product of two quark superfields, has

$$R(M) \;=\; 2R(Q) \;=\; 2\Big(1-\frac{N_c}{N_f}\Big), \qquad \Delta(M) \;=\; \frac{3}{2}\,R(M) \;=\; 3\Big(1-\frac{N_c}{N_f}\Big). \tag{I.3.74}$$

Now read off the edges,

$$\Delta(M)\big|_{N_f=\frac{3}{2}N_c} = 3\big(1-\tfrac{2}{3}\big) = 1, \qquad R(Q)\big|_{N_f=3N_c} = 1-\tfrac{N_c}{3N_c} = \tfrac{2}{3}. \tag{I.3.75}$$

At the lower edge $N_f = \frac{3}{2}N_c$ the meson has reached the free-field value $\Delta(M) = 1$, and below this edge it would violate the unitarity bound as an interacting operator. Unitarity thus diagnoses the end of the interacting electric continuation: near and below the lower edge the composite meson is becoming a free field, and the magnetic dual, where M is an elementary field with its own kinetic term, is the weakly coupled or infrared-free description that takes over. At the upper edge $N_f = 3N_c$ the quark reaches its free value $R(Q) = \frac{2}{3}$, matching the weakly coupled ultraviolet picture where asymptotic freedom is marginal. Both endpoint values (I.3.75) follow by substitution. The lower one connects the Chapter 1 unitarity bound to the SQCD window.

Exactly marginal deformations: not every conformal manifold is $\mathcal{N}=2$

A fixed point can sit on a continuous family of fixed points, a *conformal manifold*, reached by adding an exactly marginal operator. In $4d$ $\mathcal{N}=1$ the classically marginal operators are the superpotential terms of R-charge two and the gauge-kinetic term, and a combination of their couplings is exactly marginal when the corresponding beta function vanishes. The Leigh–Strassler analysis shows, in the standard examples, that this happens along a subspace cut out by the vanishing of a combination of anomalous dimensions, so the conformal manifold has dimension equal to the number of marginal couplings minus the number of independent broken global symmetries. This is genuinely $\mathcal{N}=1$ data: it is not the special geometry of an $\mathcal{N}=2$ Coulomb branch (Chapter 5), and it has its own marginality and anomaly constraints. The canonical example is $\mathcal{N}=4$ super Yang–Mills viewed as $\mathcal{N}=1$, whose three-coupling exactly marginal family contains the β-deformation. The proof that the codimension condition is exact, and the structure of these manifolds, are developed in Part III; here we only record that $\mathcal{N}=1$ fixed points come in families.

I.3.7 Seiberg duality and the conformal window

Seiberg duality relates distinct ultraviolet gauge theories that flow to the same infrared physics. The electric theory is $SU(N_c)$ with N_f fundamental flavors. The magnetic dual is $SU(N_f - N_c)$ with dual quarks $q, \widetilde{q}$, a gauge-singlet meson M (the electric composite $Q\widetilde{Q}$, now an elementary field on the magnetic side), and the cubic superpotential

$$W \;=\; M\,q\,\widetilde{q}. \tag{I.3.76}$$

These are different ultraviolet theories (different gauge groups, different matter, an extra gauge singlet), not two copies of one theory. The claim is an infrared equivalence, supported by four checks.

Check one: the continuous global symmetries match. Both frames carry $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$. On the magnetic side the dual quarks sit at $R(q) = R(\widetilde{q}) = N_c/N_f$ and the meson at $R_M = 2(1 - N_c/N_f)$, so the magnetic

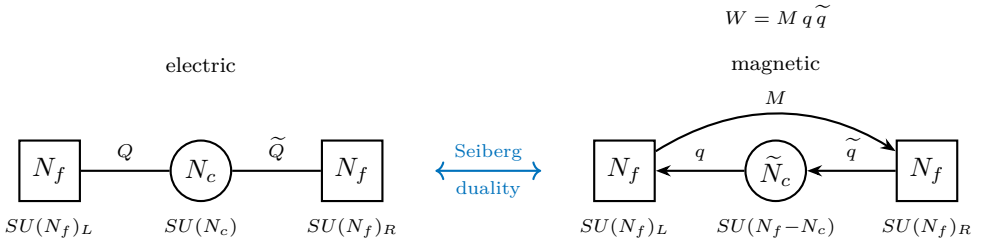


Figure I.3.1: Seiberg duality as a quiver move. Circles are gauge groups, squares are flavor groups, and arrows on the magnetic side denote a fundamental at the tail and an antifundamental at the head. The electric flavor legs are unoriented; Tables I.3.1 and I.3.3 fix their gauge and flavor representations. The electric $SU(N_c)$ theory has quarks $Q, \widetilde{Q}$ running through the gauge node. The magnetic dual has gauge group $SU(N_f - N_c)$, the dual quarks $q, \widetilde{q}$ in the representations listed in the magnetic table, an added gauge-singlet meson M (the elementary image of $Q\widetilde{Q}$, drawn as the arc joining the two flavor nodes), and the cubic superpotential $W = Mq\widetilde{q}$ (I.3.76).

superpotential $W = Mq\widetilde{q}$ carries R-charge

$$R_M + R(q) + R(\widetilde{q}) \;=\; 2\big(1 - \tfrac{N_c}{N_f}\big) + 2\tfrac{N_c}{N_f} \;=\; 2, \tag{I.3.77}$$

as a superpotential must. The dual quark q is an antifundamental of $SU(N_f)_L$ and the meson M carries one $SU(N_f)_L$ fundamental and one $SU(N_f)_R$ antifundamental index.

Check two: the gauge-invariant chiral operators match. The electric meson $Q\widetilde{Q}$ maps to the elementary magnetic singlet M, and the electric baryons map to the dual baryons. This is not only a slogan; the conserved charges match by computation. The electric baryon $B_{\rm el} \sim Q^{N_c}$ uses N_c electric quarks of baryon number $B(Q) = 1$, so $B(B_{\rm el}) = N_c$. The dual baryon $b_{\rm mag} \sim q^{N_f - N_c}$ uses $N_f - N_c$ dual quarks, whose baryon number is rescaled to $B(q) = N_c/(N_f - N_c)$ precisely so that

$$B(b_{\rm mag}) \;=\; (N_f - N_c)\,\frac{N_c}{N_f - N_c} \;=\; N_c \;=\; B(B_{\rm el}). \tag{I.3.78}$$

The R-charges match by an independent computation. The electric baryon carries $R(B_{\rm el}) = N_c\,R(Q) = N_c\big(1 - N_c/N_f\big)$, while the dual quark sits at $R(q) = 1 - (N_f - N_c)/N_f = N_c/N_f$, so

$$R(b_{\rm mag}) \;=\; (N_f - N_c)\,\frac{N_c}{N_f} \;=\; N_c\Big(1 - \frac{N_c}{N_f}\Big) \;=\; R(B_{\rm el}). \tag{I.3.79}$$

Both the baryon number and the R-charge match, each built from independent magnetic charges (the rescaled $B(q)$ and the dual-quark $R(q)$). The displayed generators have matching charges; the full identification of their chiral-ring relations is part of the duality statement.

Check three: the 't Hooft anomalies match, trace by trace. The magnetic matter content and global charges are collected in Table I.3.3, the companion to the electric Table I.3.1. The magnetic fermion content is the dual gaugino (the adjoint of $SU(N_f - N_c)$, dimension $(N_f - N_c)^2 - 1$, at $R_\psi = 1$), the $2N_f$ dual quark fermions (dimension $N_f - N_c$ each, at $R_\psi = N_c/N_f - 1$), and the N_f^2 meson fermions (at $R_\psi = R_M - 1 = 1 - 2N_c/N_f$). Run the first flavor anomaly.

field	$\widetilde{N}$ gauge	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\mathbf{1}$	$+N_c/\widetilde{N}$	N_c/N_f
$\widetilde{q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-N_c/\widetilde{N}$	N_c/N_f
M	$\mathbf{1}$	$\square$	$\overline{\square}$	0	$2(1 - N_c/N_f)$
$\widetilde{\lambda}$ (dual gaugino)	adj	$\mathbf{1}$	$\mathbf{1}$	0	$+1$

Table I.3.3: The magnetic dual matter content and global charges (writing $\widetilde{N} \equiv N_f - N_c$ for the dual gauge rank), with superpotential $W = Mq\widetilde{q}$. The meson M carries one $SU(N_f)_L$ fundamental and one $SU(N_f)_R$ antifundamental index and is gauge-singlet. The baryon charge of the dual quark is rescaled so that the dual baryon matches the electric baryon. The meson fermion ($R_\psi = 1 - 2N_c/N_f$) is the field that closes every flavor and R anomaly match below.

Under $SU(N_f)_L$, the dual quark q is an antifundamental (it carries -1 in the cubic index) with $N_f - N_c$ colors, and the meson M is a fundamental with N_f "colors" (its $SU(N_f)_R$ index runs over N_f values):

$$SU(N_f)_L^3\big|_{\rm mag} \;=\; \underbrace{-(N_f - N_c)}_{\text{dual } q} \;+\; \underbrace{N_f}_{\text{meson } M} \;=\; N_c \;=\; SU(N_f)_L^3\big|_{\rm el}. \tag{I.3.80}$$

The meson contribution is what closes the match: drop M and the magnetic side gives $-(N_f - N_c) \neq N_c$. The meson is essential. The opposite flavor group works the same way with the conjugate representations,

$$SU(N_f)_R^3\big|_{\rm mag} \;=\; \underbrace{+(N_f - N_c)}_{\text{dual } \widetilde{q}} \;-\; \underbrace{N_f}_{\text{meson } M} \;=\; -N_c \;=\; SU(N_f)_R^3\big|_{\rm el}, \tag{I.3.81}$$

the sign flipped because $\widetilde{q}$ is a fundamental and M an antifundamental of $SU(N_f)_R$. The cubic and gravitational R-traces match as exact identities in (N_c, N_f). Summing the three magnetic fermion species in Table I.3.3 (dual gaugino, $2N_f$ dual quarks, N_f^2 meson components) gives

$$\mathrm{Tr}\, R\big|_{\rm mag} = \big[\widetilde{N}^2 - 1\big] + 2N_f\widetilde{N}\Big(\tfrac{N_c}{N_f} - 1\Big) + N_f^2\Big(1 - \tfrac{2N_c}{N_f}\Big) = -(N_c^2 + 1), \tag{I.3.82}$$

$$\mathrm{Tr}\, R^3\big|_{\rm mag} = \big[\widetilde{N}^2 - 1\big] + 2N_f\widetilde{N}\Big(\tfrac{N_c}{N_f} - 1\Big)^3 + N_f^2\Big(1 - \tfrac{2N_c}{N_f}\Big)^3 = (N_c^2 - 1) - \frac{2N_c^4}{N_f^2}, \tag{I.3.83}$$

with $\widetilde{N} \equiv N_f - N_c$ the dual rank of Table I.3.3. Both sums equal the electric (I.3.41) and (I.3.42) (the algebra is a short exercise in expanding the cubes, and the meson's N_f^2 term is indispensable in (I.3.83)). The mixed trace matches as well,

$$SU(N_f)_L^2\, U(1)_R\big|_{\rm mag} \;=\; (N_f - N_c)\cdot \tfrac{1}{2}\left(\tfrac{N_c}{N_f} - 1\right) + N_f \cdot \tfrac{1}{2}\left(1 - \tfrac{2N_c}{N_f}\right) \;=\; -\frac{N_c^2}{2N_f}, \quad \text{(I.3.84)}$$

the electric value (I.3.43), again with the meson's contribution included. We have now run five explicit traces ($SU(N_f)_L^3$, $SU(N_f)_R^3$, $\mathrm{Tr}\, R$, $\mathrm{Tr}\, R^3$, and $SU(N_f)_L^2 U(1)_R$), each matching electric to magnetic. The meson is necessary for the flavor traces. Its fermion has $R_\psi = 1 - 2N_c/N_f$, so at the self-dual $N_f = 2N_c$ point its contribution to the two pure R traces vanishes. The explicit $N_c = 3, N_f = 5$ example shows the nondegenerate contributions: $N_f^2 R_\psi = -5$ and $N_f^2 R_\psi^3 = -1/5$.

Check four: dualization returns the rank. Giving one electric quark a mass, $\Delta W = m\, Q_{N_f} \widetilde{Q}_{N_f}$, removes a flavor, $N_f \to N_f - 1$, with the holomorphic scales matched by (I.3.9). On the magnetic side the mass maps (under $M = Q\widetilde{Q}$) to a linear meson term $\Delta W = m\, M_{N_f N_f}$. Its F-term $\partial W/\partial M_{N_f N_f} = q_{N_f} \widetilde{q}_{N_f} + m = 0$ forces a nonzero dual-quark vacuum expectation value, which Higgses the dual gauge group: $SU(N_f - N_c) \to SU(N_f - 1 - N_c)$, lowering the dual rank by one. The magnetic theory after Higgsing is the dual of the $N_f - 1$ electric theory, so dualizing it back returns

$$N_f - (N_f - N_c) \;=\; N_c, \quad \text{(I.3.85)}$$

the original electric rank when the flavor number is held fixed. Mass decoupling provides a separate compatibility test, since Higgsing lowers the magnetic rank to $N_f - 1 - N_c$, the dual rank of the electric theory with one flavor removed. Dualizing twice without removing a flavor returns $SU(N_f - (N_f - N_c)) = SU(N_c)$, with the meson-of-the-meson identified back as $Q\widetilde{Q}$.

The conformal window and the phases. The window of behaviors as N_f varies is the map every later chapter cites. It is collected in Table I.3.4. The conformal window, where both the electric and magnetic frames flow to the same interacting fixed point, is

$$\tfrac{3}{2}\, N_c \;<\; N_f \;<\; 3\, N_c, \quad \text{(I.3.86)}$$

with the upper edge $N_f = 3N_c$ the asymptotic-freedom boundary ($b_0 = 0$) and the lower edge $N_f = \frac{3}{2}N_c$ where the magnetic theory loses asymptotic freedom and the dual becomes free (below it, the magnetic $SU(N_f - N_c)$ is infrared-free with the meson, the "free magnetic" phase). The upper edge is $3N_c$, not $2N_c$: at $N_c = 3$ the window is $4.5 < N_f < 9$, so $N_f = 5, 6, 7, 8$ are all conformal. Below the window the behaviors are sharper, and three of them are worth working explicitly: the runaway, the quantum deformation, and s-confinement.

The Affleck–Dine–Seiberg runaway: $0 < N_f < N_c$

For $0 < N_f < N_c$ there is too little matter to lift the runaway, and a nonperturbative superpotential is generated on the meson moduli space,

$$W_{\rm ADS} \;=\; (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{1/(N_c - N_f)}, \tag{I.3.87}$$

where $M = Q\widetilde{Q}$ is the $N_f \times N_f$ meson and the overall coefficient is fixed by the symmetries up to a convention-dependent constant. The exponent is forced by holomorphy and the R-symmetry, and that determination is worth doing. The superpotential must carry R-charge two. The quark fermion R-charge here is $R_\psi(Q) = -N_c/N_f$ (the same formula (I.3.37)), so the meson scalar and its determinant carry

$$R(M) = 2\big(1 - \tfrac{N_c}{N_f}\big), \qquad R(\det M) = N_f\, R(M) = 2(N_f - N_c), \tag{I.3.88}$$

$\det M$ being a product of N_f eigenvalues. A power $(\det M)^p$ then has R-charge $2p(N_f - N_c)$, and combining with Λ (R-neutral in the anomaly-free symmetry and of mass dimension one) the unique holomorphic, R-charge-two, dimension-three combination is

$$W_{\rm ADS} \;\propto\; \Lambda^{(3N_c - N_f)/(N_c - N_f)}\,(\det M)^{-1/(N_c - N_f)}, \tag{I.3.89}$$

which is (I.3.87) up to the constant: the power $1/(N_c - N_f)$ is the unique exponent that makes $R(W) = 2$ and gives W mass dimension three. Take the F-term. On the diagonal slice $M = v\,\mathbb{1}_{N_f}$, $\det M = v^{N_f}$, so

$$\frac{\partial W_{\rm ADS}}{\partial v} \;\propto\; \frac{\partial}{\partial v}\, v^{-N_f/(N_c - N_f)} \;\propto\; v^{-N_f/(N_c - N_f) - 1}. \tag{I.3.90}$$

This is a fixed-sign power of v: it never vanishes at finite v, and it decreases monotonically as $v \to \infty$. There is no finite stationary point; the F-term is solved only as $v \to \infty$, that is $\det M \to \infty$. The correct statement is therefore not "no supersymmetric vacuum survives" but

no *finite* supersymmetric vacuum; the meson runs away to $\det M \to \infty$. (I.3.91)

Contrast a Wess–Zumino mass term, whose F-term has a genuine finite zero,

$$W = \tfrac{1}{2} m v^2 - c v \;\Longrightarrow\; \frac{\partial W}{\partial v} = m v - c = 0 \;\Longrightarrow\; v = \frac{c}{m}. \tag{I.3.92}$$

The ADS F-term (I.3.90) has no such zero; its monotone fall-off to zero at infinity is the runaway. A finite zero versus a monotone approach to zero at infinity is exactly the difference between a vacuum and a runaway.

Worked case: $SU(3)$ with $N_f = 2$. Here $3N_c - N_f = 7$ and $N_c - N_f = 1$, so the exponent $1/(N_c - N_f) = 1$ and the ADS superpotential is a clean simple pole,

$$W_{\rm ADS} \;=\; (N_c - N_f)\,\frac{\Lambda^7}{\det M} \;=\; \frac{\Lambda^7}{\det M}, \tag{I.3.93}$$

with M the 2×2 meson. On the diagonal slice $M = v\,\mathbb{1}_2$, $\det M = v^2$, so

$$W_{\rm ADS} \;\propto\; \Lambda^7 v^{-2}, \qquad \frac{\partial W_{\rm ADS}}{\partial v} \;\propto\; -2\,\Lambda^7 v^{-3}. \tag{I.3.94}$$

The F-term has no zero at finite v and falls to zero only as $v \to \infty$: the meson runs away, exactly the general pattern made concrete. (The negative exponent persists throughout $0 < N_f < N_c$, so the same finite-v obstruction applies.)

The quantum-deformed moduli space: $N_f = N_c$

At $N_f = N_c$ the matter is just enough to support a moduli space but not a runaway. Classically the mesons and baryons satisfy

$$\det M - B\widetilde{B} \;=\; 0, \tag{I.3.95}$$

an identity that follows from the epsilon contractions (M is $N_c \times N_c$ here, and $B, \widetilde{B}$ are the single baryon and antibaryon). Quantum mechanically the constraint is *deformed*,

$$\det M - B\widetilde{B} \;=\; \Lambda^{2N_c}. \tag{I.3.96}$$

Read the right-hand side. The exponent is $2N_c = b_0|_{N_f = N_c}$ (with $b_0 = 3N_c - N_f = 2N_c$), the power associated with the holomorphic instanton factor. This scaling alone does not derive the quantum deformation. The symmetry check is immediate. The anomaly-free R-charge here is $R(Q) = 1 - N_c/N_c = 0$, so $\det M$ and $B\widetilde{B}$ are both R-neutral, and Λ^{2N_c} (a pure power of the holomorphic scale) is a singlet of $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$: every term in (I.3.96) carries the same (trivial) global charges, so the deformation is allowed. The physical consequence is that the *origin is removed*: at $M = B = \widetilde{B} = 0$ the left side of (I.3.96) is zero but the right side is $\Lambda^{2N_c} \neq 0$, so the classically singular origin is not part of the quantum moduli space. Different regions of the quantum moduli space realize different symmetry-breaking patterns: mesonic regions break the chiral flavor symmetry, while the baryonic region keeps the nonabelian flavor symmetry but breaks $U(1)_B$. What the deformation forbids is a single vacuum at $M = B = \widetilde{B} = 0$ that preserves all the classical global symmetries at once.

The smallest case is fully explicit. At $N_c = N_f = 2$ the meson M is a 2×2 matrix and there is one baryon B and one antibaryon $\widetilde{B}$, so the constraint reads

$$M_{11}M_{22} - M_{12}M_{21} - B\widetilde{B} \;=\; \Lambda^4. \tag{I.3.97}$$

The relative sign can be traced directly to the antisymmetric meson of the four doublets. Fix the color tensor and the converted antiquarks by

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P_i = E^{-1}\widetilde{Q}_i = -E\widetilde{Q}_i, \qquad X = (Q_1, Q_2, P_1, P_2), \qquad V = X^t E X.$$

Here Q_i is a color column and $\widetilde{Q}_i$ is the column of antifundamental components, so $M_{ij} = Q_i^t \widetilde{Q}_j = Q_i^t E P_j$. Choose $B = Q_1^t E Q_2$ and $\widetilde{B} = \widetilde{Q}_1^t E \widetilde{Q}_2 = P_1^t E P_2$. In this

ordering the independent infrared coordinates assemble as

$$V = \begin{pmatrix} 0 & B & M_{11} & M_{12} \\ -B & 0 & M_{21} & M_{22} \\ -M_{11} & -M_{21} & 0 & \widetilde{B} \\ -M_{12} & -M_{22} & -\widetilde{B} & 0 \end{pmatrix}, \qquad \mathrm{Pf}\, V = B\widetilde{B} - \det M. \tag{I.3.98}$$

Indeed, $\mathrm{Pf}\, V = V_{12}V_{34} - V_{13}V_{24} + V_{14}V_{23}$. The enhanced-flavor form of the quantum constraint is $\mathrm{Pf}\, V = \Lambda_{\mathrm{P},2}^4$. For this doublet ordering and these baryon phases its relation to (I.3.97) is therefore

$$\Lambda_{\mathrm{P},2}^4 = -\Lambda_2^4, \qquad \Lambda_2 \equiv \Lambda \quad (N_f = 2).$$

The subscript P denotes the holomorphic scale normalized in the Pfaffian form. This is a phase of the instanton factor, not a change of its magnitude. At a common subtraction scale, writing both factors as $\mu^{b_0} e^{2\pi i \tau}$ identifies $\tau_{\mathrm{P}} = \tau + \frac{1}{2}$ modulo integers. It is a translation between two holomorphic coupling conventions for the same theory, with the operator definitions fixed above; shifting the physical theta angle while holding one convention fixed would instead change the theory. The identity $V = X^t E X$ explains the classical constraint. In the quantum relation, V denotes the independent composite coordinates, whose constraint is deformed; one must not also impose the classical rank-two identity.

Two limiting branches illustrate the deformation,

$$\underbrace{\det M = \Lambda^4}_{B=\widetilde{B}=0,\ \text{chiral symmetry broken}}, \qquad \underbrace{B\widetilde{B} = -\Lambda^4}_{M=0,\ U(1)_B\ \text{broken}}, \tag{I.3.99}$$

the meson branch ($B = \widetilde{B} = 0$) with the quarks condensed and the flavor symmetry broken to a maximal subgroup, the baryon branch ($M = 0$) preserving the nonabelian flavor symmetry. Neither branch passes through $M = B = \widetilde{B} = 0$: the quantum constraint physically separates the chiral-symmetry-breaking and baryon-number-breaking regions and forbids the symmetric origin. The explicit 2×2 determinant shows why the quantum constraint removes the origin.

S-confinement: $N_f = N_c + 1$

At $N_f = N_c + 1$ the theory confines without breaking chiral symmetry, and the infrared degrees of freedom are the smooth gauge invariants M (an $(N_c+1) \times (N_c+1)$ meson), B, and $\widetilde{B}$ (the baryon and antibaryon, now $N_c + 1$-component flavor vectors). They are governed by a dynamically generated superpotential

$$W \;=\; \frac{1}{\Lambda^{2N_c-1}} \big(B\, M\, \widetilde{B} - \det M \big), \tag{I.3.100}$$

where the prefactor exponent $2N_c - 1 = b_0|_{N_f = N_c+1}$ (with $b_0 = 3N_c - (N_c+1) = 2N_c - 1$) again carries the right instanton scaling. The F-terms of (I.3.100) reproduce the classical constraints. Differentiating with respect to the meson,

$$\frac{\partial W}{\partial M_{ij}} \;\propto\; B_i \widetilde{B}_j - (\mathrm{adj}\, M)_{ji} \;=\; 0, \tag{I.3.101}$$

where $(\mathrm{adj}\, M)_{ji}$ is the cofactor (the derivative of $\det M$); this is the classical relation $B_i\widetilde{B}_j = \mathrm{cofactor}_{ij}(M)$. The baryon F-terms give

$$\frac{\partial W}{\partial B_i} \;\propto\; (M\widetilde{B})_i = 0, \qquad \frac{\partial W}{\partial \widetilde{B}_j} \;\propto\; (BM)_j = 0. \tag{I.3.102}$$

The two terms in this superpotential have different jobs. The determinant supplies the cofactor in the meson F-term, while the baryon product supplies the other half of that relation. Dropping the $\det M$ term from (I.3.100) would leave

$$\frac{\partial W}{\partial M_{ij}}\Big|_{\det M \text{ dropped}} \;\propto\; B_i\widetilde{B}_j \;\neq\; (\mathrm{adj}\, M)_{ji}, \tag{I.3.103}$$

which is *not* the classical relation; both terms are required for the F-terms to reproduce the constraints. The two pieces $BM\widetilde{B}$ and $\det M$ are not independent decorations; together they are the smooth chiral-ring relation.

For $SU(2)$ with $N_f = 3$, the same ordering is $X = (Q_1, Q_2, Q_3, P_1, P_2, P_3)$ with $P_i = -E\widetilde{Q}_i$. Define the flavor-vector baryons by

$$B_i = \frac{1}{2}\epsilon_{ijk}Q_j^t E Q_k, \qquad \widetilde{B}_i = \frac{1}{2}\epsilon_{ijk}\widetilde{Q}_j^t E \widetilde{Q}_k, \qquad \epsilon_{123} = +1.$$

Thus $B = (Q_2^t E Q_3, Q_3^t E Q_1, Q_1^t E Q_2)$, and similarly for $\widetilde{B}$. All fifteen entries of the antisymmetric meson are now fixed:

$$V = \begin{pmatrix} 0 & B_3 & -B_2 & M_{11} & M_{12} & M_{13} \\ -B_3 & 0 & B_1 & M_{21} & M_{22} & M_{23} \\ B_2 & -B_1 & 0 & M_{31} & M_{32} & M_{33} \\ -M_{11} & -M_{21} & -M_{31} & 0 & \widetilde{B}_3 & -\widetilde{B}_2 \\ -M_{12} & -M_{22} & -M_{32} & -\widetilde{B}_3 & 0 & \widetilde{B}_1 \\ -M_{13} & -M_{23} & -M_{33} & \widetilde{B}_2 & -\widetilde{B}_1 & 0 \end{pmatrix}.$$

Expanding the Pfaffian along its first row gives

$$\begin{aligned} \mathrm{Pf}\, V &= \sum_{i,j=1}^{3} B_i M_{ij} \widetilde{B}_j - \det M, \\ -\frac{\mathrm{Pf}\, V}{\Lambda_{\mathrm{P},3}^3} &= \frac{BM\widetilde{B} - \det M}{\Lambda_3^3}, \qquad \Lambda_{\mathrm{P},3}^3 = -\Lambda_3^3. \end{aligned} \tag{I.3.104}$$

Here $\Lambda_3 \equiv \Lambda$ in (I.3.100). For example, the term $V_{12}V_{36}V_{45}$ is $B_3 M_{33}\widetilde{B}_3$ with positive sign, while the three mixed entries give $-\det M$. The same phase choice works under decoupling. Add mM_{33} and set the mixed heavy-flavor mesons and $B_{1,2}, \widetilde{B}_{1,2}$ to zero. The M_{33} equation gives

$$0 = \frac{B_3\widetilde{B}_3 - \det M_{2\times 2}}{\Lambda_3^3} + m, \qquad \det M_{2\times 2} - B_3\widetilde{B}_3 = m\Lambda_3^3 = \Lambda_2^4.$$

flavor range	infrared behavior	signature
$0 < N_f < N_c$	runaway, no finite vacuum	ADS superpotential (I.3.87), $\det M \to \infty$
$N_f = N_c$	quantum-deformed moduli space	$\det M - B\widetilde{B} = \Lambda^{2N_c}$, origin removed
$N_f = N_c + 1$	s-confinement	smooth origin, dynamical W (I.3.100), F-terms = constraints
$N_c + 1 < N_f \leq \frac{3}{2}N_c$	free magnetic	$SU(N_f - N_c)$ infrared-free with meson M
$\frac{3}{2}N_c < N_f < 3N_c$	conformal window	both frames flow to one fixed point
$N_f \geq 3N_c$	free electric / not asymptotically free	$b_0 \leq 0$, no infrared confinement scale from asymptotic freedom

Table I.3.4: The Seiberg-duality map of $SU(N_c)$ SQCD as a function of the flavor number N_f. The upper edge of the conformal window is $3N_c$, the asymptotic-freedom boundary, not $2N_c$. The four checks of the text are run trace by trace in the magnetic frame for the conformal-window entries, and the three sub-window phases (runaway, quantum deformation, s-confinement) are worked above.

The Pfaffian scales obey the same matching, $\Lambda^4_{\mathrm{P},2} = m\Lambda^3_{\mathrm{P},3}$. Both worked examples therefore use one consistent phase translation.

Common misconception: Seiberg duality is not an identity of one theory

The electric $SU(N_c)$ SQCD and the magnetic $SU(N_f - N_c)$ theory with meson M and $W = Mq\widetilde{q}$ are *different* ultraviolet theories: different gauge groups, different matter, an extra gauge singlet. They flow to the *same* infrared fixed point in the conformal window. The shared 't Hooft anomalies and chiral ring are evidence for that infrared equivalence, not a statement that the two theories are the same in the ultraviolet. Anomaly matching is necessary, not sufficient.

I.3.8 Dynamical supersymmetry breaking and the index

Supersymmetry can be broken by the dynamics, but only when the dynamics allows it, and the Witten index is the obstruction. If $\mathrm{Tr}\,(-1)^F \neq 0$, a nonzero index guarantees a zero-energy state that supersymmetry pairs nowhere, so it cannot be broken. Pure super Yang–Mills is the worked instance: its index $\mathrm{Tr}\,(-1)^F = N_c$ of (I.3.24) is nonzero, so it confines without breaking supersymmetry. Dynamical breaking therefore requires the index to vanish or to be inapplicable, which the chiral, runaway, and metastable

mechanisms arrange.

The 3-2 model, at field-content level. The canonical chiral example is the 3-2 model: gauge group $SU(3) \times SU(2)$ with one quark Q in the $(\mathbf{3}, \mathbf{2})$, antiquarks $\bar{U}, \bar{D}$ in the $(\overline{\mathbf{3}}, \mathbf{1})$, and a doublet L in the $(\mathbf{1}, \mathbf{2})$, with a tree-level superpotential $W_{\text{tree}} = h\, Q \bar{D} L$. The matter is chiral (it does not come in vector-like pairs), so there is no gauge-invariant mass term and no continuous deformation back to a vector-like theory; the Witten-index argument does not apply, and the door to breaking is open. Without the tree coupling the model has classical flat directions. The tree superpotential lifts those directions, while the $SU(3)$ dynamics (with $N_f = 2 < N_c = 3$) generates an ADS-type superpotential of the form (I.3.87); the tree and dynamical terms then compete, and the combined F-term equations have no supersymmetric solution, producing a stable, supersymmetry-breaking vacuum at a calculable scale when $h \ll 1$. The structural point is that chiral matter evades the index obstruction and the result is genuine (not metastable) breaking. The detailed F-term solution is developed in Part III.

The ISS rank condition. The cleanest worked breaking is metastable, and it is a rank condition. Take $SU(N_c)$ SQCD with massive flavors in the free magnetic range, $N_c + 1 < N_f < \frac{3}{2} N_c$. Its magnetic dual is an infrared-free $SU(N_f - N_c)$ theory with dual quarks $q, \widetilde{q}$ and a local meson Φ, an $N_f \times N_f$ matrix. We take these local fields to have canonical leading Kähler metric. Their normalization relative to the electric composite M is a separate matching input; it is not fixed by the deformation scale μ. With equal masses the local superpotential is

$$W \;=\; h\,\mathrm{Tr}\big[\Phi\,(q\widetilde{q})^T\big] \;-\; h\,\mu^2\,\mathrm{Tr}\,\Phi. \tag{I.3.105}$$

The transpose contracts the flavor representations in Table I.3.3: the cubic term is $h \sum_{i,j,a} \Phi_{ij} q_{ia} \widetilde{q}_{aj}$. Under the surviving diagonal flavor group, $\Phi \mapsto U \Phi U^\dagger$, $q \mapsto U^* q$ and $\widetilde{q} \mapsto \widetilde{q} U^T$, so this contraction is invariant. Writing $(F_\Phi)_{ij} = \partial W / \partial \Phi_{ij}$, the meson equation is

$$F_\Phi \;=\; h\,\big(q\,\widetilde{q} - \mu^2\,\mathbb{1}_{N_f}\big) \;=\; 0, \tag{I.3.106}$$

which would require the $N_f \times N_f$ bilinear $q\widetilde{q}$ to equal $\mu^2 \mathbb{1}_{N_f}$, a full-rank matrix. Write $\widetilde{N}_c = N_f - N_c$. With indices ordered as in $q\widetilde{q}$, take q to be $N_f \times \widetilde{N}_c$ and $\widetilde{q}$ to be $\widetilde{N}_c \times N_f$. Their product is $N_f \times N_f$ but has rank at most $\widetilde{N}_c$, so

$$\mathrm{rank}(q\,\widetilde{q}) \;\le\; N_f - N_c \;<\; N_f. \tag{I.3.107}$$

For nonzero h and μ^2, the product cannot reach the rank-N_f matrix $\mu^2 \mathbb{1}_{N_f}$. Its kernel has dimension at least N_c; on each orthonormal kernel vector, $q\widetilde{q} - \mu^2 \mathbb{1}_{N_f}$ acts as $-\mu^2$. Thus its squared Frobenius norm is at least $N_c |\mu^2|^2$. With the canonical leading Kähler metric, this is a lower bound on the meson F-term energy. It is attained by

$$q\widetilde{q} = \mathrm{diag}(\underbrace{\mu^2, \ldots, \mu^2}_{N_f - N_c}, \underbrace{0, \ldots, 0}_{N_c}) \implies F_\Phi = -h\mu^2 \text{ on the last } N_c \text{ entries}, \tag{I.3.108}$$

with $N_f - N_c$ satisfied entries. At $\Phi = 0$, the quark F-terms vanish; choosing $q = \mu(\mathbb{1}_{\widetilde{N}_c}, 0)^T$ and $\widetilde{q} = \mu(\mathbb{1}_{\widetilde{N}_c}, 0)$ also sets the gauge D-terms to zero. The leading local

minimum is therefore

$$V_{\min} \;=\; \sum_{\text{unsatisfied}} |F_\Phi|^2 \;=\; N_c\,|h\,\mu^2|^2 \;>\; 0, \tag{I.3.109}$$

strictly positive in this local description, establishing supersymmetry breaking at the ISS metastable vacuum. Supersymmetric vacua elsewhere in field space are discussed below. This is the rank-condition mechanism: a full-rank F-term equation cannot be solved by a lower-rank bilinear, and the rank deficit N_c counts the unsatisfied F-terms, each contributing $|h\mu^2|^2$ to the vacuum energy. (The would-be Goldstino sits in the $\mathrm{Tr}\,\Phi$ direction whose F-term (I.3.106) is the one that cannot be cancelled.)

The breaking here is metastable, not absolute. Supersymmetric vacua do reappear elsewhere in field space, restored nonperturbatively where the meson runs out to large values, but they are separated from the rank-condition vacuum by a barrier and it is parametrically long-lived. The rank condition explains the local breaking, while the index criterion shows why a nonzero index would have forbidden supersymmetry breaking everywhere.

At full rank the obstruction disappears,

$$N_c = 0 \implies N_f - N_c = N_f \implies \text{rank deficit } = N_f - (N_f - N_c) = 0, \tag{I.3.110}$$

so the F-term (I.3.106) *is* solvable and supersymmetry is unbroken (no gauge group). Were the bilinear allowed full rank, there would be no breaking; it is the geometry of rectangular matrices, the rank deficit $N_c > 0$, that forces the breaking.

The supersymmetric index, named. The Witten index has a refined cousin, the supersymmetric (or superconformal) index on $S^3 \times S^1$. It is a protected counting partition function $\mathrm{Tr}\,(-1)^F x^{\cdots}$ over states annihilated by a chosen supercharge: it counts short multiplets with signs, is invariant under continuous deformations, and matches across Seiberg duals, a refinement of the 't Hooft anomalies. We name its role here; Part III develops the elliptic-gamma-function integral and the duality-invariance analysis. No index is computed here beyond the Witten count.

Exit checklist

After this chapter the reader can

1. identify the multiplets and chiral ring of a $4d$ $\mathcal{N} = 1$ theory, read the dynamical scale off $b_0 = 3N_c - \sum_i T(r_i)$, and match scales across a threshold by $\Lambda_{N_f-1}^{3N_c-(N_f-1)} = m\,\Lambda_{N_f}^{3N_c-N_f}$;
2. compute pure-SYM gaugino condensation, the $\mathbb{Z}_{2N_c} \to \mathbb{Z}_2$ breaking, the N_c vacua, and the Witten index $\mathrm{Tr}\,(-1)^F = N_c$;
3. assign a trial R-symmetry, apply the fermion shift $R_\psi = R_s - 1$, derive $R(Q) = 1 - N_c/N_f$ from gauge-R anomaly freedom, and compute the 't Hooft traces $\mathrm{Tr}\,R = -(N_c^2+1)$, $\mathrm{Tr}\,R^3 = (N_c^2-1) - 2N_c^4/N_f^2$, $SU(N_f)_L^3 = +N_c$, $SU(N_f)_R^3 = -N_c$;

4. run a-maximization to the superconformal R, getting $a(3,6) = 123/64$, $c(3,6) = 163/64$ for SQCD, the irrational extremum for adjoint SQCD (a root of the *quadratic* $da/dR_X = 0$), and the symmetry-forced $R(X) = \frac{2}{3}$ for the dP_0 quiver, and remember that it is a, not c, that is maximized;

5. write the Seiberg dual of SQCD, verify the four checks (including the magnetic $SU(N_f)_L^3 = -(N_f - N_c) + N_f = N_c$ with the meson essential), work the ADS runaway, the quantum-deformed constraint $\det M - B\widetilde{B} = \Lambda^{2N_c}$, and the s-confining superpotential, and place a theory in the window map of Table I.3.4;

6. apply the Witten-index criterion for dynamical supersymmetry breaking, work the ISS rank condition $\mathrm{rank}(q\widetilde{q}) \leq N_f - N_c < N_f$, and state which deep theorems (a-maximization, Seiberg duality, the superconformal index) are stated here and developed elsewhere.

Status and sources

The chapter uses its stated superpotential and a-maximization conventions and the gauge-group and quiver conventions of the dP_0 example. Line operators can distinguish gauge groups with the same Lie algebra.

The index sum (I.3.3), electric trace sums (I.3.41)–(I.3.42), conditional magnetic trace match (I.3.82)–(I.3.83), and local ISS rank energy (I.3.109) are mathematical deductions [MATH] under the stated field-theory inputs. Such calculations do not prove holomorphic decoupling, condensation, duality, phase existence, ISS stabilization, or index identities.

§I.3.1 Multiplets, holomorphy, scale, threshold. The chiral/vector multiplets, the holomorphic τ (I.3.1), the one-loop coefficient $b_0 = 3N_c - \sum_i T(r_i)$ (I.3.2) ($= 3N_c - N_f$ for SQCD, (I.3.3)), the RG-invariant scale $\Lambda^{b_0} = \mu^{b_0} e^{2\pi i \tau}$ (I.3.7), and the threshold matching $\Lambda_{N_f-1}^{3N_c-(N_f-1)} = m\,\Lambda_{N_f}^{3N_c-N_f}$ (I.3.9) (the exponent shifts by the removed Dynkin index $2T(\square) = 1$); the gauge-group-versus-algebra aside. **Sources:** (Tachikawa (2018) the holomorphy/Λ spine). Additional calculations include b_0 as a Dynkin-index sum, the $b_0 = 0$ edge at $N_f = 3N_c$, the RG-invariance of Λ, and the threshold-matching exponent/identity.

§I.3.2 The chiral ring and non-renormalization. The chiral ring as gauge-invariant chiral words modulo $\partial W = 0$, the meson (I.3.14) and baryon (I.3.15) generators and their gradings; the non-renormalization of W and the one-loop-exact holomorphic gauge coupling. **Sources:** (Intriligator and Seiberg (1996) the SQCD chiral-ring and non-renormalization review). Additional calculations include the chiral-ring grading and the meson generator on SQCD and on the dP_0 quiver.

§I.3.3 Pure super Yang–Mills. The anomalous $U(1)_R \to \mathbb{Z}_{2N_c}$ (I.3.20), gaugino condensation $\langle\lambda\lambda\rangle_k \sim \Lambda^3 e^{2\pi i k/N_c}$ (I.3.21) breaking $\mathbb{Z}_{2N_c} \to \mathbb{Z}_2$, the N_c vacua, the Witten index $\mathrm{Tr}\,(-1)^F = N_c$ (I.3.24), and the optional Veneziano–Yankielowicz W_{VY} (I.3.25) recovering $S^{N_c} \sim \Lambda^{3N_c}$; the condensate is unbroken-SUSY zero-energy vacua, not a SUSY-breaking order parameter; the carefully stated scale-matching pointer to the $4d$ $\mathcal{N} = 2$ chapter (the $\mathcal{N} = 2$ Seiberg–Witten scale is $\mathcal{N} = 2$ transmutation, related to this condensate only after an $\mathcal{N} = 1$ mass deformation). **Sources:** (Witten (1982) the index; Tachikawa (2018) the condensate / vacuum

count). Additional calculations include $\mathrm{Tr}\,(-1)^F = N_c$ from the assumed gaugino-condensate vacuum count; this is no independent index computation. The VY stationary equation and root count are derived from the assumed effective superpotential. The counting argument for the index supplies neither the potential nor its stationary polynomial. The assumed effective superpotential is sourced to Veneziano and Yankielowicz (1982).

§I.3.4 R-symmetry, NSVZ, the superconformal R. The fermion R-rule $R_\psi = R_s - 1$ (I.3.30) (gaugino $R = 1$), the full NSVZ beta function (I.3.31) and its numerator condition (I.3.32), the ABJ $SU(N_c)^2$-$U(1)_R = 0$ condition (I.3.35) solving $R(Q) = 1 - N_c/N_f$ (I.3.37), and the superconformal R as the constrained abelian combination with $\Delta = \frac{3}{2}R$ (back to Chapter 1). **Sources:** the fermion shift appears in (I.2.23). For both the chiral fermion shift and gaugino assignment, Martin (1998, §4.11, eqs. (4.11.1), (4.11.5), (4.11.6), Table 4.1) fixes $R(\theta) = 1$, $R_\psi = R_\phi - 1$ and $R_\lambda = 1$. Novikov et al. (1983) gives the NSVZ beta function. Under the fermion R shift and field charges, the arithmetic gives the trace inputs and the ABJ condition solving $R(Q) = 1 - N_c/N_f$.

§I.3.5 't Hooft matching and the Konishi anomaly. 't Hooft anomalies as renormalization-group invariants (necessary, not sufficient, for a duality, a necessary condition), the electric SQCD table (Table I.3.1) with the declared $SU(N_f)_R$ antifundamental convention, the L/R-split cubic traces $SU(N_f)_L^3 = +N_c$, $SU(N_f)_R^3 = -N_c$ (I.3.40), the explicitly run $\mathrm{Tr}\,R = -(N_c^2+1)$ (I.3.41), $\mathrm{Tr}\,R^3 = (N_c^2-1) - 2N_c^4/N_f^2$ (I.3.42), $SU(N_f)_L^2 U(1)_R = -N_c^2/(2N_f)$ (I.3.43), and the Konishi anomaly (I.3.47) with its coefficient equal to the gauge index feeding the ABJ trace. **Sources:** ('t Hooft (1980) anomaly matching; Konishi (1984) the Konishi anomaly). Under the representations, R assignment and index normalization, the arithmetic gives the SQCD trace arithmetic with the L/R split and the common Konishi/ABJ index coefficient ($T(\square) = \frac{1}{2}$), and the $\mathrm{Tr}\,R^3$ calculation.

§I.3.6 a-maximization. The central charges a, c (I.3.49) as fermion traces over a trial R, the local-maximum selection of the superconformal R (it is a, not c; the constraints are gauge-anomaly freedom and $R(W) = 2$, the 't Hooft anomalies being the data, not vanishing constraints); the SQCD rational functions (I.3.53) and the numeric $a(3,6) = 123/64$, $c(3,6) = 163/64$ (I.3.55) with $a \neq c$; the genuine adjoint-SQCD extremum (the trial $a(R_X)$ cubic so $da/dR_X = 0$ is *quadratic*; the maximizing root irrational, Hessian < 0, c not maximized); and the dP_0 symmetric $R(X) = \frac{2}{3}$ (I.3.65)–(I.3.66); the Banks–Zaks fixed point (the weakly coupled entry to the conformal window at $N_f = 3N_c - \epsilon N_c$) is the point where perturbation theory controls the fixed point and checks the exact anomaly/a-maximization R. **Sources:** (Intriligator and Wecht (2003) a-maximization; Banks and Zaks (1982) the fixed point; Lawrence et al. (1998) the orbifold-daughter quiver). Additional calculations include the SQCD a, c reassembled from the traces with $a \neq c$ and the numeric $123/64, 163/64$; the adjoint-SQCD maximization (interior, irrational, $a'' < 0$, anomaly-free, a-max $\neq$ c-max with $c(a\text{-max}) < c(c\text{-max})$, and a cubic / da quadratic, N1-9); the dP_0 common charge fixed at $R(X) = \frac{2}{3}$ by $R(W) = 2$; and the Banks–Zaks two-loop zero positive and $O(\epsilon)$ inside the window. The Hessian-sign discriminator ($a'' < 0$ is a maximum) is the anti-vacuousness check.

§I.3.7 Seiberg duality and the conformal window. The magnetic dual $SU(N_f - N_c)$ with meson M and $W = Mq\widetilde{q}$ (I.3.76); the four checks (global symmetries, chiral operators, 't Hooft anomalies with the explicit magnetic $SU(N_f)_L^3 = -(N_f - N_c) + N_f = N_c$ (I.3.80) and the matched R-traces (I.3.82), the meson essential, fixed-N_f rank involution (I.3.85), and separate mass-decoupling compatibility); the conformal window $\frac{3}{2}N_c < N_f < 3N_c$ (I.3.86); the worked sub-window phases, the ADS runaway (I.3.87)–(I.3.91) (no finite vacuum, $\det M \to \infty$, F-term (I.3.90)), the quantum-deformed constraint (I.3.96) (origin removed), the s-confining superpotential (I.3.100) and its F-terms (I.3.101)–(I.3.102); and the full flavor map (Table I.3.4).

Sources: (Seiberg (1995) the duality + window; Affleck et al. (1984) the ADS superpotential). Under the proposed magnetic field content, the arithmetic gives the electric↔magnetic 't Hooft match trace by trace with the L/R split and the meson essential, the re-dualization involution back to N_c, the window bounds $(\frac{3}{2}N_c, 3N_c)$ for the stated conformal window, the ADS F-term runaway conditional on the ADS superpotential with no finite zero, the quantum-deformed origin removal, and the s-confinement F-terms reproducing the classical constraints.

§I.3.8 Dynamical SUSY breaking and the index. The Witten-index obstruction $(\mathrm{Tr}\,(-1)^F \neq 0 \Rightarrow$ unbroken), worked on pure SYM; the ISS rank-condition fixture (I.3.105)–(I.3.107) (the F-term $F_\Phi = h(q\widetilde{q} - \mu^2 \mathbb{1}_{N_f})$ unsolvable because $\mathrm{rank}(q\widetilde{q}) \leq N_f - N_c < N_f$, metastable breaking, SUSY vacua restored nonperturbatively); the 3-2 model named at statement level; the superconformal index named and routed to Part III. **Sources:** (Affleck et al. (1985) the 3-2 model; Intriligator, Seiberg, and Shih (2006), §2, equations (2.3)–(2.5), the canonical local potential; its quantum stabilization and distant vacua are separate dynamical inputs). The flavor contraction follows Seiberg (1995), §3, equations (3.1)–(3.3), translated to the matrix shapes above. The example with real rational matrices and $N_c = 5, N_f = 7, h = 2, \mu^2 = 4$ illustrates why the energy alone cannot distinguish a transposed contraction. In the local ISS rank condition, $F_\Phi = 0$ would require rank N_f. The rank deficit leaves N_c unsatisfied F-terms and gives positive vacuum energy.

Routed (stated, not proved). The a-maximization / a-theorem / superconformal-R theorem and the accidental-symmetry correction; the Seiberg-duality theorem and the full quantum moduli spaces; the superconformal index computation, the elliptic-gamma integral, and its duality invariance; and the gauge global-form / line-operator classification. All are stated and cited here; their proofs live in Part III.

Further reading

The quantum-moduli constraint at $N_f = N_c$ is given in Seiberg (1994, §4, eq. (4.1)); the $N_f = N_c + 1$ s-confining superpotential and its classical constraints are in Seiberg (1994, §5, eq. (5.4), §2, eq. (2.9)). For $SU(2)$ those source formulas are specifically $\mathrm{Pf}\, V = \Lambda^4_{\mathrm{P},2}$ and $W = -\mathrm{Pf}\, V/\Lambda^3_{\mathrm{P},3}$. Equations (I.3.98) and (I.3.104) derive their component signs from the declared ordering and baryon definitions; the local phase bridge is $\Lambda^{6-N_f}_{\mathrm{P},N_f} = -\Lambda^{6-N_f}_{N_f}$ for $N_f = 2, 3$. The component assembly and Pfaffian expansions establish the normalization translation and its mass-decoupling consequence under the cited nonperturbative premises. The general $SU(N_c)$ electric/magnetic superconformal-index identity is treated in Dolan and Osborn (2009, §6); Romelsberger (2006) gives the earlier $SU(2)$, three-flavor comparison. The exact holomorphic structure of $4d$ $\mathcal{N} = 1$ gauge theories is developed in the lectures Intriligator and Seiberg (1996); exactly marginal deformations and superconformal manifolds in Leigh and Strassler (1995); the glueball effective superpotential in Veneziano and Yankielowicz (1982). The central charges a, c and a-maximization are treated in Anselmi et al. (1998). Dynamical supersymmetry breaking, including the metastable-vacua mechanism, is reviewed in Intriligator and Seiberg (2007). The superconformal index that counts protected operators is introduced in Romelsberger (2006) and Kinney et al. (2007) and developed in Dolan and Osborn (2009).

For modern entry points, Gadde (2022) gives a graduate-level introduction to the superconformal index, Smilga (2023) reviews Witten-index calculations across four-

dimensional supersymmetric gauge theories, and Bajeot et al. (2023) illustrates how the classical s-confinement and duality toolkit continues to generate new $\mathcal{N} = 1$ dynamics.

References

't Hooft, G. (1980). "Naturalness, Chiral Symmetry, and Spontaneous Chiral Symmetry Breaking". In: *Recent Developments in Gauge Theories*. Vol. 59, pp. 135–157. DOI: 10.1007/978-1-4684-7571-5_9.

Affleck, I., M. Dine, and N. Seiberg (1984). "Dynamical Supersymmetry Breaking in Supersymmetric QCD". In: *Nucl. Phys. B* 241, pp. 493–534. DOI: 10.1016/0550-3213(84)90061-0.

Affleck, I., M. Dine, and N. Seiberg (1985). "Dynamical Supersymmetry Breaking in Four-Dimensions and Its Phenomenological Implications". In: *Nucl. Phys. B* 256, pp. 557–599. DOI: 10.1016/0550-3213(85)90408-0.

Anselmi, D. et al. (1998). "Nonperturbative formulas for central functions of supersymmetric gauge theories". In: *Nucl. Phys. B* 526, pp. 543–571. DOI: 10.1016/S0550-3213(98)00278-8. arXiv: hep-th/9708042.

Bajeot, S., S. Benvenuti, and M. Sacchi (2023). "S-confining gauge theories and supersymmetry enhancements". In: *JHEP* 08, p. 042. DOI: 10.1007/JHEP08(2023)042. arXiv: 2305.10274 [hep-th].

Banks, T. and A. Zaks (1982). "On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions". In: *Nucl. Phys. B* 196, pp. 189–204. DOI: 10.1016/0550-3213(82)90035-9.

Dolan, F. A. and H. Osborn (2009). "Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories". In: *Nucl. Phys. B* 818, pp. 137–178. DOI: 10.1016/j.nuclphysb.2009.01.028. arXiv: 0801.4947 [hep-th].

Gadde, A. (2022). "Lectures on the Superconformal Index". In: *J. Phys. A* 55.6, p. 063001. DOI: 10.1088/1751-8121/ac42ac. arXiv: 2006.13630 [hep-th].

Intriligator, K. A. and N. Seiberg (1996). "Lectures on supersymmetric gauge theories and electric-magnetic duality". In: *Nucl. Phys. B Proc. Suppl.* 45BC, pp. 1–28. DOI: 10.1016/0920-5632(95)00626-5. arXiv: hep-th/9509066.

Intriligator, K. A. and N. Seiberg (2007). "Lectures on Supersymmetry Breaking". In: *Class. Quant. Grav.* 24. Ed. by C. Bachas et al., S741–S772. DOI: 10.1088/0264-9381/24/21/S02. arXiv: hep-ph/0702069.

Intriligator, K. A., N. Seiberg, and D. Shih (2006). "Dynamical SUSY breaking in meta-stable vacua". In: *JHEP* 04, p. 021. DOI: 10.1088/1126-6708/2006/04/021. arXiv: hep-th/0602239.

Intriligator, K. A. and B. Wecht (2003). "The Exact superconformal R symmetry maximizes a". In: *Nucl. Phys. B* 667, pp. 183–200. DOI: 10.1016/S0550-3213(03)00459-0. arXiv: hep-th/0304128.

Kinney, J. et al. (2007). "An Index for 4 dimensional super conformal theories". In: *Commun. Math. Phys.* 275, pp. 209–254. DOI: 10.1007/s00220-007-0258-7. arXiv: hep-th/0510251.

Konishi, K. (1984). "Anomalous Supersymmetry Transformation of Some Composite Operators in SQCD". In: *Phys. Lett. B* 135, pp. 439–444. DOI: 10.1016/0370-2693(84)90311-3.

Lawrence, A. E., N. Nekrasov, and C. Vafa (1998). "On conformal field theories in four-dimensions". In: *Nucl. Phys. B* 533, pp. 199–209. DOI: 10.1016/S0550-3213(98)00495-7. arXiv: hep-th/9803015.

Leigh, R. G. and M. J. Strassler (1995). "Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory". In: *Nucl. Phys. B* 447, pp. 95–136. DOI: 10.1016/0550-3213(95)00261-P. arXiv: hep-th/9503121.

Martin, S. P. (1998). "A Supersymmetry primer". In: *Adv. Ser. Direct. High Energy Phys.* 18. Ed. by G. L. Kane, pp. 1–98. DOI: 10.1142/9789812839657_0001. arXiv: hep-ph/9709356.

Novikov, V. A. et al. (1983). "Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculus". In: *Nucl. Phys. B* 229, pp. 381–393. DOI: 10.1016/0550-3213(83)90338-3.

Romelsberger, C. (2006). "Counting chiral primaries in N = 1, d=4 superconformal field theories". In: *Nucl. Phys. B* 747, pp. 329–353. DOI: 10.1016/j.nuclphysb.2006.03.037. arXiv: hep-th/0510060.

Seiberg, N. (1994). "Exact Results on the Space of Vacua of Four Dimensional SUSY Gauge Theories". In: *Phys. Rev. D* 49, pp. 6857–6863. DOI: 10.1103/PhysRevD.49.6857. arXiv: hep-th/9402044.

Seiberg, N. (1995). "Electric - magnetic duality in supersymmetric nonAbelian gauge theories". In: *Nucl. Phys. B* 435, pp. 129–146. DOI: 10.1016/0550-3213(94)00023-8. arXiv: hep-th/9411149.

Smilga, A. V. (2023). "Witten index in 4D supersymmetric gauge theories". In: *Int. J. Mod. Phys. A* 38.31, p. 2330015. DOI: 10.1142/S0217751X23300156. arXiv: 2308.12941 [hep-th].

Tachikawa, Y. (2018). "Lectures on 4d N=1 dynamics and related topics". In: arXiv: 1812.08946 [hep-th].

Veneziano, G. and S. Yankielowicz (1982). "An Effective Lagrangian for the Pure N=1 Supersymmetric Yang-Mills Theory". In: *Phys. Lett. B* 113, p. 231. DOI: 10.1016/0370-2693(82)90828-0.

Witten, E. (1982). "Constraints on Supersymmetry Breaking". In: *Nucl. Phys. B* 202, p. 253. DOI: 10.1016/0550-3213(82)90071-2.

Chapter I.4

$2d$ supersymmetric field theories

Reader contract. In two dimensions a supercharge has a definite worldsheet chirality. Keeping both chiralities gives $\mathcal{N} = (2,2)$ supersymmetry, whose chiral and twisted-chiral sectors distinguish complex from Kähler data. Keeping only the right-moving supercharges gives $\mathcal{N} = (0,2)$ supersymmetry. Its unpaired left-moving fermions carry independent E and J interactions and make anomaly conditions unavoidable. The two theories meet in the elliptic genus, which retains protected information along the renormalization-group flow.

The chapter develops these connections through calculations. The one-loop twisted superpotential gives the quantum-cohomology ring of $\mathbb{CP}^{N-1}$; the quintic gauge theory exhibits geometric and Landau–Ginzburg regimes. Euler homogeneity enforces E/J closure in its gauged $\mathcal{N} = (0,2)$ description. Fermion anomalies determine the equations for c-extremization, while an explicit rank orbit shows the order-three action of triality. The chiral ring and trial R-symmetry of Chapter 3 provide the four-dimensional comparison. Deeper phase, duality, and index results enter here through their stated hypotheses and working consequences.

What this chapter delivers

The $2d$ SUSY kinematics: the (p,q) count, worldsheet chirality, $\mathcal{N} = (2,2)$ versus $\mathcal{N} = (0,2)$, and the R-symmetries $U(1)_V$, $U(1)_A$ (§I.4.1). Part I, the $\mathcal{N} = (2,2)$ world: the two holomorphic sectors W versus $\widetilde{W}$ (§I.4.2); the gauged linear sigma model, the $\mathbb{CP}^{N-1}$ quantum cohomology $\mathbb{C}[\Sigma]/(\Sigma^N - q)$, and the quintic phases (§I.4.3); the A/B twists and the Hori–Vafa mirror (§I.4.4). Part II, the $\mathcal{N} = (0,2)$ world: the E/J scalar potential and closure $\sum_a \mathrm{Tr}(E_a J^a) = 0$ (§I.4.5); bundles and the gauged $\mathcal{N} = (2,2) \to \mathcal{N} = (0,2)$ E-term (§I.4.6); anomalies and c-extremization to $c_R = 3k/(k+2)$ (§I.4.7); and triality (§I.4.8). Part III, the shared interface: the elliptic genus (§I.4.9) and the $\mathcal{N} = (2,2)$-versus-$\mathcal{N} = (0,2)$ comparison (§I.4.10).

I.4.1 $2d$ kinematics and R-symmetries

Everything special about two dimensions starts from one fact: a spinor in two dimensions can be Majorana–Weyl. A Majorana–Weyl spinor carries a single real component and a definite worldsheet chirality, left-moving or right-moving. A supercharge inherits that chirality, and so the amount of supersymmetry cannot be quoted by a single number as in four dimensions. It is a pair (p, q) with p left-moving and q right-moving real supercharges, and the total count is the sum

$$\#\,\text{supercharges} \;=\; p + q, \tag{I.4.1}$$

the $2d$ Majorana–Weyl reality row of Chapter 1. This chapter treats the two ends of the four-supercharge family. The maximal case is $\mathcal{N} = (2, 2)$,

$$\mathcal{N} = (2,2): \quad p = 2,\ q = 2, \qquad \#\,\text{supercharges} \;=\; 4, \tag{I.4.2}$$

the dimensional reduction of $4d$ $\mathcal{N} = 1$ on a two-torus: the four real supercharges survive, now sorted into two right-moving and two left-moving,

$$4d\ \mathcal{N} = 1 \;\xrightarrow{T^2}\; 2d\ \mathcal{N} = (2,2), \qquad 4 = 2 + 2, \tag{I.4.3}$$

the four-supercharge row that reads $4d\ \mathcal{N} = 1 \equiv 3d\ \mathcal{N} = 2 \equiv 2d\ \mathcal{N} = (2, 2)$ on the Chapter 1 relation. The minimal case is $\mathcal{N} = (0, 2)$,

$$\mathcal{N} = (0,2): \quad p = 0,\ q = 2, \qquad \#\,\text{supercharges} \;=\; 2, \tag{I.4.4}$$

two real supercharges, all right-moving. This is the chiral, holomorphic half of $\mathcal{N} = (2, 2)$: it keeps the two right-moving supercharges and discards the two left-moving ones. It has no four-dimensional Lorentz-invariant parent, since there is no way to sort four-dimensional supercharges into right-movers and left-movers, so its structures are the genuinely two-dimensional part of the chapter.

The chapter map. The split by chirality organizes the chapter. Part I is the $\mathcal{N} = (2, 2)$ world, four balanced supercharges, Kähler and complex geometry: the two holomorphic sectors (§I.4.2), the gauged linear sigma model (§I.4.3), the A/B twists and mirror symmetry (§I.4.4). Part II is the $\mathcal{N} = (0, 2)$ world, the chiral half with two right-movers and no left-movers, a holomorphic bundle over a target: E/J data (§I.4.5), bundles and the gauged E-term (§I.4.6), anomalies and c-extremization (§I.4.7), triality (§I.4.8). Part III is the shared interface: the elliptic genus (§I.4.9), a $\mathcal{N} = (0, 2)$ object that specializes to $\mathcal{N} = (2, 2)$, and a comparison table (§I.4.10). Each structure sits in the world it belongs to: mirror symmetry and the twists are $\mathcal{N} = (2, 2)$-only, triality and c-extremization are $\mathcal{N} = (0, 2)$-central, the elliptic genus is genuinely shared.

The two R-symmetries. Why should two dimensions carry *two* abelian R-symmetries where four dimensions carries one? Because the two right-moving and two left-moving supercharges can be rotated by two commuting phases, not one. A $\mathcal{N} = (2, 2)$ theory has an R-symmetry group that contains two commuting $U(1)$ factors, the *vector* R-symmetry $U(1)_V$ and the *axial* R-symmetry $U(1)_A$. The vector R-symmetry acts

with the same sign on the left- and right-moving supercharges; the axial R-symmetry acts with opposite signs. Writing the two supercharge phases as $Q_\pm \to e^{i\alpha_\pm} Q_\pm$ (the subscript is the worldsheet chirality), the two combinations are

$$U(1)_V:\ \alpha_+ = \alpha_- = \alpha_V, \qquad U(1)_A:\ \alpha_+ = -\alpha_- = \alpha_A, \tag{I.4.5}$$

the vector rotation the diagonal, the axial the anti-diagonal. This distinction has a physical consequence. The two R-symmetries distinguish the two topological twists of §I.4.4 (A uses $U(1)_V$, B uses $U(1)_A$), and the axial one is the fragile factor that can be anomalous. Its anomaly is proportional to the first Chern class of a sigma-model target, so $U(1)_A$ survives precisely on a Calabi–Yau, which is why the B-twist (and only the B-twist) is gated by the Calabi–Yau condition, and why the sigma-model coupling can run when only $U(1)_V$ survives.

The two sectors carry opposite axial charge. Under $U(1)_V$ a chiral Φ and a twisted-chiral Σ both carry their vector R-charge, and $W(\Phi)$, $\widetilde{W}(\Sigma)$ must each be $U(1)_V$-neutral as a superspace integral. Under $U(1)_A$ the chiral and twisted-chiral fields carry opposite sign, because the axial rotation distinguishes D_+ from D_-. So the two rings of §I.4.2 are graded by different R-symmetries, the algebraic root of mirror symmetry, which exchanges the two R-symmetries and so the two sectors,

$$U(1)_V \ \leftrightarrow\ U(1)_A, \qquad \Phi \ \leftrightarrow\ \Sigma, \qquad W \ \leftrightarrow\ \widetilde{W}. \tag{I.4.6}$$

Carry away one fact: two dimensions has two abelian R-symmetries because it has two holomorphic sectors, and each sector is graded by the R-symmetry its superspace measure respects.

The multiplet vocabulary. The multiplets are graded by which combinations of the supersymmetry covariant derivatives annihilate them. Three multiplets carry the $\mathcal{N} = (2,2)$ content.

- *Chiral* Φ, annihilated by both barred derivatives $\bar{D}_\pm \Phi = 0$. Content: a complex scalar ϕ, a Dirac fermion (one right-mover ψ_+ and one left-mover ψ_-), and an auxiliary field. It is the reduction of the $4d$ $\mathcal{N} = 1$ chiral multiplet of Chapter 2, and it carries the ordinary superpotential $W(\Phi)$.

- *Twisted-chiral* Σ, annihilated by the *opposite* pair $\bar{D}_+ \Sigma = D_- \Sigma = 0$. It has the same field content as a chiral (a complex scalar, a Dirac fermion, an auxiliary) and the opposite constraint. It has no four-dimensional analog. It is indispensable because the field strength of a $\mathcal{N} = (2,2)$ vector, $\Sigma = \bar{D}_+ D_- V$, is twisted-chiral, and its holomorphic self-coupling is the *twisted* superpotential $\widetilde{W}(\Sigma)$, developed in §I.4.2.

- *Vector* V, carrying the gauge field, a Dirac gaugino, and the auxiliary D-field. In two dimensions the gauge field has no propagating polarization, so the physical content of the vector sits in its twisted-chiral field strength Σ.

The two chiralities are the two constraints

$$\text{chiral}:\ \bar{D}_\pm \Phi = 0, \qquad \text{twisted-chiral}:\ \bar{D}_+ \Sigma = D_- \Sigma = 0, \qquad \Sigma = \bar{D}_+ D_- V, \tag{I.4.7}$$

the opposite pairing $\bar{D}_+, D_-$ marking the twisted sector. The $\mathcal{N} = (0,2)$ algebra has one barred right-moving derivative $\bar{D}_+$, and its three multiplets are graded by worldsheet chirality.

- *Chiral* Φ, annihilated by $\bar{D}_+$. Content: a complex scalar ϕ and a single right-moving Weyl fermion ψ_+, nothing left-moving.
- *Fermi* Λ, the genuinely new object. Content: a single left-moving Weyl fermion λ_- and *no* propagating scalar. It is only constrained chiral, $\bar{D}_+\Lambda = E$, with E a holomorphic function of the chiral fields; $E = 0$ is a free Fermi multiplet, and a nonzero E is one of the two pieces of $\mathcal{N} = (0,2)$ interaction data of §I.4.5.
- *Vector* V, carrying the gauge field and a left-moving gaugino.

The two chiral $\mathcal{N} = (0,2)$ constraints are

$$\text{chiral}: \ \bar{D}_+\Phi = 0, \qquad \text{Fermi}: \ \bar{D}_+\Lambda = E(\Phi), \tag{I.4.8}$$

the Fermi only constrained chiral, its right-hand side E the first piece of $\mathcal{N} = (0,2)$ interaction data. The Fermi multiplet is what makes two dimensions chiral: it supplies a left-moving fermion with no bosonic partner, so a generic $\mathcal{N} = (0,2)$ spectrum has a right-minus-left imbalance with no analog in four dimensions.

The $\mathcal{N} = (2,2) \to \mathcal{N} = (0,2)$ decomposition. Every $\mathcal{N} = (2,2)$ multiplet is a sum of $\mathcal{N} = (0,2)$ multiplets, and their field content provides a direct first check. Count the content of a $\mathcal{N} = (2,2)$ chiral: one complex scalar (two real bosonic fields), one right-moving fermion, and one left-moving fermion, written $(\text{boson}, \text{right}, \text{left}) = (2,1,1)$. A $\mathcal{N} = (0,2)$ chiral has content $(2,1,0)$: the scalar and the right-mover, no left-mover. A $\mathcal{N} = (0,2)$ Fermi has content $(0,0,1)$: the missing left-mover, nothing else. So

$$\begin{aligned} \mathcal{N} = (2,2) \text{ chiral } &= \mathcal{N} = (0,2) \text{ chiral } + \mathcal{N} = (0,2) \text{ Fermi}, \\ (2,1,1) &= (2,1,0) + (0,0,1), \end{aligned} \tag{I.4.9}$$

component by component. The Fermi is not optional in the decomposition: dropping it would leave the $\mathcal{N} = (2,2)$ chiral without its left-moving fermion, breaking the count. Likewise a $\mathcal{N} = (2,2)$ vector splits as

$$\mathcal{N} = (2,2) \text{ vector } = \mathcal{N} = (0,2) \text{ vector } + \mathcal{N} = (0,2) \text{ adjoint chiral}, \tag{I.4.10}$$

the $\mathcal{N} = (0,2)$ vector carrying the gauge field and the left-moving gaugino, the adjoint $\mathcal{N} = (0,2)$ chiral carrying the complex scalar and right-moving fermion. This is how any $\mathcal{N} = (2,2)$ theory is read as a $\mathcal{N} = (0,2)$ theory, and it is exactly why the anomaly cancellations of §I.4.7 are automatic for $\mathcal{N} = (2,2)$ theories: each $\mathcal{N} = (2,2)$ chiral brings a $\mathcal{N} = (0,2)$ chiral and a $\mathcal{N} = (0,2)$ Fermi in matched pairs, so their chiral contributions cancel term by term. Table I.4.1 collects the content.

supersymmetry	multiplet	complex scalar	right fermion ψ_+	left fermion λ_-
$\mathcal{N}=(2,2)$	chiral Φ	1	1	1
$\mathcal{N}=(2,2)$	twisted-chiral Σ	1	1	1
$\mathcal{N}=(2,2)$	vector V $(+A_\mu)$	1	1	1
$\mathcal{N}=(0,2)$	chiral Φ	1	1	0
$\mathcal{N}=(0,2)$	Fermi Λ	0	0	1
$\mathcal{N}=(0,2)$	vector V	0	0	1 (gaugino)

Table I.4.1: The $2d$ multiplet content by worldsheet chirality. A $\mathcal{N}=(2,2)$ chiral $=\mathcal{N}=(0,2)$ chiral $+\mathcal{N}=(0,2)$ Fermi (the right-fermion goes with the scalar, the left-fermion is the Fermi); a $\mathcal{N}=(2,2)$ vector $=\mathcal{N}=(0,2)$ vector $+\mathcal{N}=(0,2)$ adjoint chiral, and it also carries the gauge field A_μ, which has no propagating polarization in $2d$. The twisted-chiral Σ is the gauge-invariant field strength of the $\mathcal{N}=(2,2)$ vector: same field content as a chiral, opposite constraint $\bar{D}_+\Sigma = D_-\Sigma = 0$, and it carries the complexified Fayet–Iliopoulos and theta parameter $t = r - i\theta$ in its twisted superpotential.

Common misconception: $\mathcal{N}=(0,2)$ is "the $\mathcal{N}=2$ of two dimensions"

The pair $\mathcal{N}=(0,2)$ names *two* supercharges, not four. Reading it as "$\mathcal{N}=2$" by analogy with four dimensions, and so expecting four supercharges, confuses it with $\mathcal{N}=(2,2)$, which is the genuine four-supercharge algebra. The two indices are the left-moving and right-moving counts, so $\mathcal{N}=(0,2)$ is two right-movers and zero left-movers; the total is $p+q=2$. Likewise $(4,4)$, with eight supercharges, is a different and larger algebra, outside this chapter. The label counts the two chiralities separately precisely because $2d$ supersymmetry is graded by worldsheet chirality.

Part I. The $\mathcal{N}=(2,2)$ world

The next three sections build $\mathcal{N}=(2,2)$ dynamics as one coherent world: its two holomorphic sectors and rings (§I.4.2), the gauged linear sigma model that turns those data into Calabi–Yau geometry (§I.4.3), and the two topological twists with mirror symmetry that compute the two rings and exchange them (§I.4.4). All of it is $\mathcal{N}=(2,2)$-specific; the $\mathcal{N}=(0,2)$ world of Part II keeps only the pieces that survive discarding two supercharges.

I.4.2 The $\mathcal{N} = (2,2)$ holomorphic sectors and their rings

A $4d$ $\mathcal{N} = 1$ theory has one holomorphic self-coupling, the superpotential. A $2d$ $\mathcal{N} = (2,2)$ theory has two, and understanding why is the first genuinely two-dimensional lesson. The reason is the twisted-chiral multiplet: because the two right-moving and two left-moving supercharges combine in two inequivalent ways, there are two kinds of holomorphic constraint a multiplet can satisfy, and each supports its own superspace integral.

Two holomorphic sectors. A chiral field Φ obeys $\bar{D}_{\pm}\Phi = 0$ and couples through the ordinary superpotential, a holomorphic function integrated over half of superspace,

$$W(\Phi), \qquad \int d^2\theta\, W(\Phi) + \text{c.c.}, \tag{I.4.11}$$

exactly as in four dimensions. A twisted-chiral field Σ obeys the opposite constraint $\bar{D}_{+}\Sigma = D_{-}\Sigma = 0$ and couples through the *twisted* superpotential, a holomorphic function integrated over the *other* half of superspace,

$$\widetilde{W}(\Sigma), \qquad \int d\theta^{+} d\bar{\theta}^{-}\, \widetilde{W}(\Sigma) + \text{c.c.} \tag{I.4.12}$$

The two constraints and the two measures are genuinely separate data. Holding W and $\widetilde{W}$ apart is the whole reason for the twisted-chiral distinction: a $\mathcal{N} = (2,2)$ gauge theory carries *both* an ordinary superpotential for its matter and a twisted superpotential for its field strength, and they never mix.

FI and theta as twisted-chiral data. Here is the canonical use of the twisted sector. The field strength of a $\mathcal{N} = (2,2)$ vector is the twisted-chiral $\Sigma = \bar{D}_{+}D_{-}V$, and its lowest allowed self-coupling is the *linear* twisted superpotential

$$\widetilde{W}(\Sigma) \;=\; t\,\Sigma, \qquad t \;=\; r - i\,\theta. \tag{I.4.13}$$

The single coefficient t packages two real parameters. Expanding $\int d\theta^{+} d\bar{\theta}^{-}\, t\,\Sigma$ into components, the real part gives $r\,D$ (the Fayet–Iliopoulos coupling to the auxiliary D-field), the imaginary part θ times the field strength,

$$r \;=\; \mathrm{Re}\, t \quad \text{(FI)}, \qquad \theta \;=\; -\mathrm{Im}\, t \quad \text{(theta)}. \tag{I.4.14}$$

So the FI parameter and theta angle are the coefficient of the lowest twisted superpotential, and t is the Kähler modulus labelling the chambers of the gauged linear sigma model in §I.4.3. Because the field strength is twisted-chiral, its parameters sit in $\widetilde{W}$, not W.

Two rings. The two holomorphic sectors have two chiral rings, computed with the two nilpotent supercharges. The *chiral ring* is the cohomology of the chiral supercharge, the gauge-invariant chiral operators modulo the F-term relations $\partial W = 0$; the B-twist of §I.4.4 computes it, and for a Landau–Ginzburg model $W(\Phi)$ it is the Jacobi ring

$$\mathcal{R}_B \;=\; \mathbb{C}[\Phi_i] \,/\, \big(\partial_i W\big). \tag{I.4.15}$$

The *twisted-chiral ring* is the cohomology of the other supercharge; the A-twist computes it, and for a gauge theory it is the quantum-cohomology ring generated by Σ, worked for $\mathbb{CP}^{N-1}$ in §I.4.3. Mirror symmetry exchanges the two rings, swapping W with $\widetilde{W}$ and complex-structure with Kähler data. That one theory carries both rings, computed by two twists and swapped by mirror symmetry, is the structural heart of $2d\ \mathcal{N}=(2,2)$ physics.

A worked chiral ring. The chiral ring is concrete, so it is worth computing once. Take the A_2 Landau–Ginzburg model, one chiral field Φ with superpotential $W=\Phi^3$. The Jacobi ring (I.4.15) is $\mathbb{C}[\Phi]/(\partial_\Phi W)$ with $\partial_\Phi W = 3\Phi^2$, so

$$\mathcal{R}_B \;=\; \mathbb{C}[\Phi]\,/\,(\Phi^2) \;=\; \{1,\ \Phi\}, \tag{I.4.16}$$

a two-dimensional ring with basis the identity and Φ (the field Φ^2 is set to zero by the F-term relation). Its dimension counts the ground states, two, and its grading by the R-charge $R(\Phi)=1/3$ reproduces the two chiral primaries of the A_2 minimal model. For the general A_{k+1} model $W=\Phi^{k+2}$ the Jacobi ring is

$$\mathcal{R}_B \;=\; \mathbb{C}[\Phi]\,/\,(\Phi^{k+1}) \;=\; \{1,\Phi,\dots,\Phi^{k}\}, \qquad \dim \mathcal{R}_B \;=\; k+1, \tag{I.4.17}$$

the $k+1$ chiral primaries. This is the ring the B-twist computes, and its central charge

$$\hat{c} \;=\; 1-2R(\Phi) \;=\; \frac{k}{k+2} \tag{I.4.18}$$

is exactly the $\hat{c}$ that reappears in the c-extremization of the same model in §I.4.7, where $c_R = 3\hat{c} = 3k/(k+2)$. The two computations (the Jacobi ring here, the anomaly trace there) are two views of one fixed point, and their agreement is a check the reader can run by hand.

Common misconception: a $\mathcal{N}=(2,2)$ theory has one superpotential

Carrying the four-dimensional picture wholesale into two dimensions, one expects a single holomorphic superpotential. A $\mathcal{N}=(2,2)$ theory carries *two* holomorphic data: the ordinary superpotential $W(\Phi)$ for chiral fields, integrated over $\int d^2\theta$, and the twisted superpotential $\widetilde{W}(\Sigma)$ for twisted-chiral fields, integrated over $\int d\theta^+ d\bar{\theta}^-$. They are constrained independently and never mix. The Fayet–Iliopoulos and theta parameters are twisted-chiral data, the coefficient $t = r - i\theta$ of the linear $\widetilde{W} = t\,\Sigma$; they do *not* live in the ordinary superpotential. Correspondingly there are two rings (chiral and twisted-chiral), computed by the B- and A-twists and exchanged by mirror symmetry.

I.4.3 The gauged linear sigma model, from $\mathbf{CP}^{N-1}$ to the quintic

The workhorse construction of $2d\ \mathcal{N}=(2,2)$ theories is the gauged linear sigma model. It is an abelian (or product-unitary) gauge theory whose low-energy physics computes geometry: in one regime it is a nonlinear sigma model on a Calabi–Yau, in another a

Landau–Ginzburg orbifold, and the quantum cohomology of a projective target falls out of a single one-loop calculation on its Coulomb branch. We lead with the simplest such calculation, $\mathbb{CP}^{N-1}$, because it teaches twisted-chiral fields, the FI/theta parameter, quantum cohomology, the vacuum count, and the A-model at once; then the quintic as the flagship Calabi–Yau.

The construction. A $\mathcal{N}=(2,2)$ gauged linear sigma model is a $U(1)^k$ (or $\prod_i U(N_i)$) gauge theory with chiral matter of integer charges Q_i^a, an FI/theta parameter $t_a = r_a - i\theta_a$ for each gauge factor, and a gauge-invariant superpotential W. Its bosonic potential has two pieces. The D-term sets the moment map of the gauge action equal to the FI parameter,

$$D_a \;=\; \sum_i Q_i^a\,|\phi_i|^2 \;-\; r_a \;=\; 0, \tag{I.4.19}$$

one equation per gauge factor, and the F-terms impose $\partial W/\partial\phi_i = 0$. The charge matrix Q_i^a is the same integer data a toric variety is built from on the geometry side, and the low-energy physics depends on the sign chamber of the FI parameters.

Worked instance: $\mathbf{CP}^{N-1}$ quantum cohomology from the effective twisted superpotential

Take the simplest gauged linear sigma model with no superpotential: one $U(1)$ with N chiral fields $\phi_1,\dots,\phi_N$ of charge $+1$, and FI/theta parameter $t = r - i\theta$. In the $r \gg 0$ chamber the D-term $\sum_i |\phi_i|^2 = r$ modulo the $U(1)$ phase is the projective space $\mathbb{CP}^{N-1}$, and the theory flows to the sigma model on it. Classically the cohomology ring of $\mathbb{CP}^{N-1}$ is

$$H^*(\mathbb{CP}^{N-1}) \;=\; \mathbb{C}[x]\,/\,(x^N), \qquad x^N \;=\; 0, \tag{I.4.20}$$

with x the hyperplane class, $x^N = 0$ because $\mathbb{CP}^{N-1}$ has complex dimension $N-1$. The gauge theory computes the *quantum* correction, and the computation is one loop on the Coulomb branch, where the twisted-chiral field strength Σ takes a large value.

Give Σ a large expectation value. The coupling of a charged chiral to Σ is a *twisted mass*,

$$m_i \;=\; Q_i\,\Sigma \;=\; \Sigma \quad (\text{charge } +1), \tag{I.4.21}$$

so each of the N fields gets mass Σ and can be integrated out at large Σ. This generates a one-loop contribution to the effective twisted superpotential (a massive charged field runs in a loop and shifts the holomorphic coupling, as in four dimensions); the one-loop integral of a single field of twisted mass Σ is

$$\delta\widetilde{W}_{\text{eff}} \;=\; -\,\Sigma\big(\log\Sigma - 1\big), \tag{I.4.22}$$

the standard $\Sigma\log\Sigma$ running of a $2d$ twisted mass, in units of the renormalization scale. We use this one-loop result as input; the calculation below differentiates it and does not evaluate the determinant or independently fix its sign. Summing the N identical contributions and adding the classical FI term $-t\,\Sigma$ gives

$$\widetilde{W}_{\text{eff}}(\Sigma) \;=\; -\,t\,\Sigma \;-\; N\,\Sigma\big(\log\Sigma - 1\big), \tag{I.4.23}$$

the second term the sum of the N one-loop contributions (one per charged chiral). The Coulomb-branch vacua are the critical points of $\widetilde{W}_{\text{eff}}$. Differentiating, the $\Sigma\cdot(1/\Sigma)$ piece of $\partial_\Sigma[\Sigma(\log\Sigma-1)]$ cancels against the -1, leaving

$$\frac{\partial\widetilde{W}_{\text{eff}}}{\partial\Sigma} \;=\; -\,t \;-\; N\,\log\Sigma \;=\; 0 \quad\Longrightarrow\quad \log\Sigma=-\frac{t}{N} \quad\Longrightarrow\quad \Sigma^N \;=\; e^{-t} \;=:\; q. \tag{I.4.24}$$

The vacuum equation is $\Sigma^N=q$ with $q=e^{-t}$ the instanton (Kähler) parameter, and it has N solutions, the N Nth roots of q: the theory has N isolated massive vacua on the Coulomb branch. Reading (I.4.24) as a ring relation, the twisted-chiral ring generated by Σ is

$$\mathcal{R}_A\big(\mathbb{CP}^{N-1}\big) \;=\; \mathbb{C}[\Sigma]\,/\,\big(\Sigma^N-q\big), \tag{I.4.25}$$

the *quantum cohomology* of $\mathbb{CP}^{N-1}$: the classical ring (I.4.20) with $x^N=0$ deformed to $x^N=q$, the worldsheet instanton lifting the top power of the hyperplane class from zero to q. As $q\to 0$ ($r\to+\infty$) it degenerates to the classical cohomology, of dimension N either way, matching $\dim H^*(\mathbb{CP}^{N-1})=N$. One calculation has delivered the twisted-chiral Σ, the parameter t that sets q, the quantum cohomology ring, the vacuum count N, and the A-model data, all from one effective twisted superpotential.

The smallest case makes the numbers concrete. For $N=2$, the sigma model on $\mathbb{P}^1$, the vacuum equation (I.4.24) is $\Sigma^2=e^{-t}=q$, so the two vacua sit at

$$\Sigma \;=\; \pm\,q^{1/2}, \qquad \mathcal{R}_A(\mathbb{P}^1) \;=\; \mathbb{C}[x]\,/\,(x^2-q). \tag{I.4.26}$$

The classical $H^*(\mathbb{P}^1)=\mathbb{C}[x]/(x^2)$ has $x^2=0$; the instanton correction lifts it to $x^2=q$, the degree-one rational curve contributing $q=e^{-t}$. The two massive vacua are the two supersymmetric ground states, the Witten index

$$\mathrm{Tr}(-1)^F \;=\; 2 \;=\; \chi(\mathbb{P}^1), \tag{I.4.27}$$

the Euler characteristic, and the ring has dimension two, matching $\dim H^*(\mathbb{P}^1)=2$. This is the vacuum ring and index of $\mathbb{P}^1$ from the imported effective potential, also described by the A-twist of §I.4.4.

Common misconception: the effective twisted superpotential sign is $+N\Sigma(\log\Sigma-1)$

The one-loop term in (I.4.23) carries a *minus* sign, $-N\Sigma(\log\Sigma-1)$. This sign is not a convention one is free to flip: it is fixed by the requirement that the vacuum equation give $\Sigma^N=e^{-t}=q$ with q the instanton factor. The opposite sign $+N\Sigma(\log\Sigma-1)$ would give $\Sigma^N=e^{+t}$, inconsistent both with the instanton weight $q=e^{-t}$ and with the Hori–Vafa mirror of §I.4.4. The minus sign therefore agrees with both the instanton weight and the mirror vacuum equation.

The two phases and one RG story. Now add a superpotential and let the target be Calabi–Yau, so the gauged linear sigma model flows to a conformal field theory. The low-energy physics depends on the FI chamber, and the two chambers realize two very different-looking descriptions of the *same* infrared theory. For $r\gg 0$ the D-term forces

some positively-charged field nonzero, the origin is excluded, and setting the D-term to r and dividing by the gauge group is the symplectic reduction

$$X_r \;=\; \big\{\, \phi : \textstyle\sum_i Q_i^a |\phi_i|^2 = r_a \,\big\} \big/ \prod_a U(1)_a, \tag{I.4.28}$$

equivalently the geometric-invariant-theory (GIT) quotient, a toric variety whose fan is read from the charge matrix. The F-terms of W then cut a hypersurface (or complete intersection) inside X_r: this is the *geometric* phase, a nonlinear sigma model on a Calabi–Yau. For $r \ll 0$ the same D-term forces a negatively-charged field nonzero instead, Higgsing the gauge group to a finite subgroup and leaving the remaining fields as a Landau–Ginzburg orbifold with superpotential W: the *Landau–Ginzburg* phase.

The sign of r gives the classical chambers, but the relation between them depends on the complex parameter t. The FI and theta parameters complexify into $t = r - i\theta$, ranging over a cylinder, the *quantum Kähler moduli space* whose two ends are the classical chambers. That moduli space carries a *discriminant locus*, the codimension-one set where the theory becomes singular (a noncompact Coulomb branch opens and a new massless state appears); it is one point for one-parameter models but a genuine subvariety in general. The gauged linear sigma model therefore connects the geometric and Landau–Ginzburg limits within a family of infrared superconformal theories. A path that avoids the discriminant can connect the two limits without encountering a singular theory. The classical r-axis alone does not establish that path. The full analysis of the secondary fan and analytic continuation belongs to Volume IV; the quintic shows how the limiting descriptions arise (Figure I.4.1).

The discriminant identifies where this continuation can fail. On the Coulomb branch (large Σ) the effective twisted superpotential of a Calabi–Yau gauged linear sigma model generically has no critical point at finite Σ: the D-term lifts it. At a special t on the discriminant a would-be massive vacuum becomes massless, a critical point runs off to $\Sigma \to \infty$, and a new massless degree of freedom appears in the infrared. That is the singular locus: not a thermodynamic phase boundary but a point where the low-energy theory degenerates because a massless state (a noncompact Coulomb branch) appears. A generic theta angle steers the straight-line path in t around it, so the two descriptions are connected by a path along which the theory never degenerates. This is "no phase transition": the connection runs through the quantum moduli space, avoiding the discriminant, a statement about where the theory is singular, not about the sign of classical r.

The Calabi–Yau condition on the geometric phase is a condition on the charges. The axial R-symmetry $U(1)_A$ is non-anomalous, equivalently the target has vanishing first Chern class, precisely when

$$\sum_i Q_i^a \;=\; 0 \tag{I.4.29}$$

for each gauge factor. When (I.4.29) fails the sigma model still exists, but the FI parameter runs (the target is Fano or of general type) and it is not the conformal phase this section exemplifies.

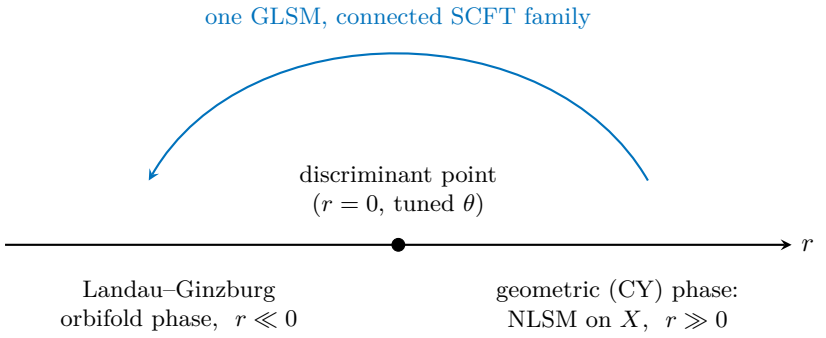


Figure I.4.1: The two phases of a Calabi–Yau gauged linear sigma model as the two ends of the quantum Kähler moduli space, the $t = r - i\theta$ cylinder. For $r \gg 0$ the theory is a nonlinear sigma model on the Calabi–Yau X; for $r \ll 0$ it is a Landau–Ginzburg orbifold. These are two limiting regimes of one gauged linear sigma model within a connected family of infrared theories. The classical $r = 0$ chamber wall contains a quantum discriminant point (drawn schematically at the origin; its location depends on conventions), where a noncompact Coulomb branch opens and a massless state appears); a generic theta angle steers the path around it, but the smoothness of the interpolation is a statement about the quantum moduli space, not the classical r-axis.

Worked instance: the quintic

The canonical Calabi–Yau example is the quintic. Take $U(1)$ with five chiral fields $\phi_1, \ldots, \phi_5$ of charge $+1$ and one field P of charge -5, with superpotential

$$W \;=\; P\, G_5(\phi), \tag{I.4.30}$$

G_5 a generic degree-five homogeneous polynomial in the five ϕ_i. The Calabi–Yau condition (I.4.29) holds,

$$\sum_i Q_i^a \;=\; 5\cdot(+1) + (-5) \;=\; 0, \tag{I.4.31}$$

so $U(1)_A$ is non-anomalous and the theory flows to a conformal fixed point. Now read the two chambers. In the $r \gg 0$ chamber the D-term is $\sum_i |\phi_i|^2 - 5|P|^2 = r > 0$, and the F-terms read

$$\frac{\partial W}{\partial P} = G_5(\phi) = 0, \qquad \frac{\partial W}{\partial \phi_i} = P\,\partial_i G_5 = 0 \;\Rightarrow\; P = 0 \tag{I.4.32}$$

(G_5 generic). With $P = 0$ the D-term reduces to $\sum_i |\phi_i|^2 = r$ modulo the $U(1)$ phase, the projective space

$$\{\textstyle\sum_i |\phi_i|^2 = r\}/U(1) \;=\; \mathbb{P}^4, \qquad \dim_{\mathbb{C}} \mathbb{P}^4 \;=\; 5 - 1 \;=\; 4, \tag{I.4.33}$$

and the F-term $G_5 = 0$ cuts a degree-five hypersurface inside $\mathbb{P}^4$, of complex dimension

$$\dim_{\mathbb{C}}\{G_5 = 0\} \;=\; 4 - 1 \;=\; 3, \tag{I.4.34}$$

the quintic Calabi–Yau threefold, with central charge $\hat{c} = \dim_{\mathbb{C}} X = 3$. The construction is the GIT quotient (I.4.33) *and then* the F-term cut, not the naive vanishing locus of W among all six fields: one equation in six variables would give the wrong dimension. Getting the two steps in the right order is the working discipline of the geometric phase.

In the $r \ll 0$ chamber the D-term $\sum_i |\phi_i|^2 - 5|P|^2 = r < 0$ can only be solved with $P \neq 0$, at $|P|^2 = -r/5$. A nonzero P Higgses the gauge group down to the stabilizer of a charge--5 field,

$$U(1) \;\longrightarrow\; \mathbb{Z}_5, \tag{I.4.35}$$

the fifth roots of unity. The five ϕ_i are then a Landau–Ginzburg model with superpotential $W = G_5(\phi)$ (with P at its VEV) orbifolded by this $\mathbb{Z}_5$. Its central charge is read from the R-charges, fixed by quasi-homogeneity of W: for W to be a good superpotential its superspace integral must be neutral, so W carries R-charge one; since $W = G_5$ is degree five and homogeneous, each field carries R-charge q_i with $5\,q_i = 1$, that is $q_i = 1/d = 1/5$ for a degree-d superpotential. A single Landau–Ginzburg field of R-charge q contributes $1 - 2q$ to the central charge (the standard $\mathcal{N} = 2$ minimal-model formula: a free chiral has $q = 0$ and $\hat{c} = 1$, and the superpotential lowers it), so the Landau–Ginzburg central charge is the sum

$$\hat{c} \;=\; \sum_i (1 - 2q_i). \tag{I.4.36}$$

For the quintic, five fields each with $q_i = 1/5$,

$$\hat{c} \;=\; 5\left(1 - \tfrac{2}{5}\right) \;=\; 5 \cdot \tfrac{3}{5} \;=\; 3, \tag{I.4.37}$$

exactly the geometric $\hat{c} = 3$. The two phases share their central charge, as they must if they are two regimes of one renormalization-group flow. The general pattern is transparent from (I.4.36): a degree-d polynomial in d fields gives

$$\hat{c} \;=\; d\left(1 - \frac{2}{d}\right) \;=\; d - 2, \tag{I.4.38}$$

which is $\dim_{\mathbb{C}}$ of the degree-d hypersurface in $\mathbb{P}^{d-1}$, a Calabi–Yau $(d-2)$-fold. The same gauged linear sigma model, with d charge-$+1$ fields and one charge-$-d$ field, produces each member,

$$d = 3:\ \hat{c} = 1 \text{ (elliptic curve)}, \qquad d = 4:\ \hat{c} = 2 \text{ (K3)}, \qquad d = 5:\ \hat{c} = 3 \text{ (quintic)}, \tag{I.4.39}$$

the charge sum $d \cdot (+1) + (-d) = 0$ ensuring the Calabi–Yau condition in every case, and the Landau–Ginzburg $\hat{c}$ matching the geometric dimension throughout.

Common misconception: the two phases are different theories

The geometric ($r \gg 0$) and Landau–Ginzburg ($r \ll 0$) phases are limiting regimes of one gauged linear sigma model within a connected family of infrared theories in the Calabi–Yau case. The geometric phase is the GIT (symplectic) quotient $\{\sum_i Q_i^a |\phi_i|^2 = r_a\}/\text{gauge}$ (a toric variety), further cut by the F-terms of W, *not* the naive vanishing locus of W alone (which would give the wrong dimension). The interpolation between the two ends is smooth in the *quantum Kähler moduli space* (the $t = r - i\theta$ cylinder), away from its discriminant locus (where a noncompact Coulomb branch opens and a massless state appears); do not read the classical r-axis by itself as proving a globally smooth interpolation. And the phases exist for any FI chamber: $\sum_i Q_i^a = 0$ is the Calabi–Yau, conformal, anomaly-free condition of the intended example, not a prerequisite for a phase to exist.

I.4.4 The A and B topological twists and mirror symmetry

A $\mathcal{N} = (2,2)$ theory admits two inequivalent topological twists, and they compute the two rings of §I.4.2. This is $\mathcal{N} = (2,2)$-specific technology, placed here immediately after the gauged linear sigma model whose rings it computes, not among the shared tools of Part III. The twist mixes the Lorentz rotation with an R-symmetry, turning one supercharge into a scalar nilpotent operator whose cohomology is the topological ring. Which R-symmetry is used distinguishes the two models, the $U(1)_V$ versus $U(1)_A$ distinction of §I.4.1.

The A-twist: quantum cohomology. The A-twist uses the *vector* R-symmetry $U(1)_V$. The A-model localizes on holomorphic maps to the target (the worldsheet instantons) and computes the quantum Kähler data: Gromov–Witten invariants and the quantum-cohomology ring. For $\mathbb{CP}^{N-1}$ it is exactly the ring of §I.4.3,

$$\mathcal{R}_A\big(\mathbb{CP}^{N-1}\big) \;=\; \mathbb{C}[x] \,/\, \big(x^N - q\big), \qquad q = e^{-t}, \tag{I.4.40}$$

the classical $x^N = 0$ deformed to $x^N = q$ by instantons. The A-twist is the twisted-chiral ring at work, and because $U(1)_V$ is non-anomalous for a broad class of sigma models it exists *generally*: it does *not* require a Calabi–Yau target. One can A-twist a Fano target and get sensible quantum cohomology.

The B-twist: the Jacobi ring. The B-twist uses the *axial* R-symmetry $U(1)_A$. The B-model localizes on constant maps and computes the classical complex-structure data (periods and variations of Hodge structure), an instanton-free holomorphic answer, the chiral ring of §I.4.2. For a Landau–Ginzburg model $W(\Phi)$ it is the Jacobi ring

$$\mathcal{R}_B \;=\; \mathbb{C}[\Phi_i] \,/\, \big(\partial_i W\big), \tag{I.4.41}$$

the F-term relations of W as a quotient. But $U(1)_A$ is anomalous in general: its anomaly is controlled by the first Chern class, so the B-twist requires

$$c_1(TX) \;=\; 0, \tag{I.4.42}$$

the Calabi–Yau condition. For a gauged linear sigma model this is the charge condition $\sum_i Q_i = 0$ of (I.4.29), because the $U(1)_A$ anomaly is proportional to the sum of the gauge charges,

$$U(1)_A \text{ anomaly } \propto \sum_i Q_i \ \propto \ c_1(TX). \tag{I.4.43}$$

This gating is asymmetric,

$$(+1)^5, -5: \ \textstyle\sum_i Q_i = 0 \text{ (B-twist } \checkmark\text{)}, \qquad (+1)^5, -4: \ \textstyle\sum_i Q_i = 1 \neq 0 \text{ (no B-twist)}, \tag{I.4.44}$$

the non-Calabi–Yau assignment $(+1)^5, -4$ having anomalous $U(1)_A$, yet it still admits the A-twist because $U(1)_V$ is non-anomalous regardless. The A-twist is not gated by the Calabi–Yau condition; the B-twist is.

Mirror symmetry and the Hori–Vafa construction

Mirror symmetry is a duality between two full $\mathcal{N} = (2,2)$ theories, not a property of their twists. A $\mathcal{N} = (2,2)$ sigma model on X is physically equivalent, as an untwisted theory, to a $\mathcal{N} = (2,2)$ theory on a mirror $\hat{X}$; for a gauged linear sigma model the mirror is the Hori–Vafa dual Landau–Ginzburg / Toda theory, and needs no twist. The construction is at field level: dualize each charged chiral Φ_i to a periodic twisted-chiral Y_i, real part $|\phi_i|^2$, shift symmetry the phase of Φ_i. The dual is a Landau–Ginzburg model of the Y_i with mirror twisted superpotential

$$\widetilde{W}(Y) \ = \ \sum_i e^{-Y_i} \ + \ \sum_a \Sigma_a \Big(\sum_i Q_i^a \, Y_i - t_a \Big), \tag{I.4.45}$$

the Σ_a enforcing the charge constraints $\sum_i Q_i^a Y_i = t_a$. For the $\mathbb{P}^{N-1}$ sigma model (one $U(1)$, N fields of charge $+1$) the mirror is

$$\widetilde{W} \ = \ \sum_{i=1}^{N} e^{-Y_i}, \qquad \sum_{i=1}^{N} Y_i = t, \tag{I.4.46}$$

whose N critical points $e^{-Y_i} = e^{-t/N}$ reproduce the N ring generators, recovering the quantum cohomology

$$\mathbb{C}[x] \,/\, (x^N - e^{-t}) \tag{I.4.47}$$

of $\mathbb{P}^{N-1}$ from the *mirror* side, matching (I.4.40). The Kähler parameter t of X becomes the complex-structure parameter of the mirror: the same t cylinder carries the mirror map.

The twists are how the duality is most sharply *used*. In the protected topological sector the A-twist of X and the B-twist of $\hat{X}$ compute the same correlators, so mirror symmetry exchanges the two models,

$$A\text{-model}(X) \ \cong \ B\text{-model}(\hat{X}), \tag{I.4.48}$$

trading the quantum Kähler data of X for the classical complex-structure data of $\hat{X}$, and so $\mathcal{R}_A(X)$ for $\mathcal{R}_B(\hat{X})$. The full topological-correlator machinery (Frobenius /

quantum-cohomology structure, variation of Hodge structure) and the mirror theorem are developed in Volume IV; here we construct the untwisted Hori–Vafa mirror, name the two twists, tie them to the two rings, and gate the B-twist by the Calabi–Yau condition.

Part II. The $\mathcal{N} = (0,2)$ world

The next four sections build $\mathcal{N} = (0,2)$ dynamics as the second coherent world. Discarding the two left-moving supercharges of $\mathcal{N} = (2,2)$ frees the left-moving fermions from their partners, and the consequences run through the whole part: two independent holomorphic data (E, J) per Fermi and a scalar potential built from them (§I.4.5); a holomorphic bundle over the target, seen sharply as a gauged $\mathcal{N} = (2,2) \to \mathcal{N} = (0,2)$ deformation (§I.4.6); a gauge anomaly that must cancel, a gravitational anomaly that only matches, and the c-extremization principle (§I.4.7); and an order-three triality (§I.4.8).

I.4.5 $\mathcal{N} = (0,2)$ Fermi multiplets, the E/J data, and the scalar potential

Now the genuinely two-dimensional world: two right-moving supercharges, no $\mathcal{N} = (2,2)$ completion. A $\mathcal{N} = (0,2)$ theory has no single superpotential but two holomorphic data per Fermi multiplet. Why those data are not free, and what they do to the scalar potential, is the central physics of the $\mathcal{N} = (0,2)$ world. The Fermi constraint and its superpotential coupling show why both functions are needed.

The two data. Each Fermi multiplet Λ_a is only constrained chiral,

$$\bar{D}_+ \Lambda_a \;=\; E_a(\Phi), \tag{I.4.49}$$

with E_a a holomorphic function of the $\mathcal{N} = (0,2)$ chiral fields (itself a $\mathcal{N} = (0,2)$ chiral superfield). The second function enters the $\mathcal{N} = (0,2)$ superpotential, a Fermi-linear integral over half of the $\mathcal{N} = (2,2)$ measure,

$$W_{\mathcal{N}=(0,2)} \;=\; \int d\theta^+ \, \sum_a \Lambda_a \, J^a(\Phi) \;+\; \text{c.c.}, \tag{I.4.50}$$

with J^a a second holomorphic function of the chiral fields. The pair (E_a, J^a) is two independent holomorphic data; neither is more fundamental, and they enter the physics symmetrically, as the scalar potential and the closure condition below make precise.

The scalar potential. The physical content of E and J is that they are the two halves of the $\mathcal{N} = (0,2)$ scalar potential. Component-expanding the Fermi kinetic term $|\bar{D}_+ \Lambda_a|^2 = |E_a|^2 + \cdots$ and the superpotential (I.4.50) generates the bosonic potential

$$V \;\supset\; \sum_a \big|E_a(\phi)\big|^2 \;+\; \sum_a \big|J^a(\phi)\big|^2 \;+\; V_D, \tag{I.4.51}$$

with V_D the gauge D-term potential. Thus both functions influence the energy of a field configuration. The supersymmetric vacua are the zeros of V,

$$E_a(\phi) \;=\; 0, \qquad J^a(\phi) \;=\; 0, \qquad \text{(plus the D-term equations)}, \tag{I.4.52}$$

for every a: the $E_a = 0$ and $J^a = 0$ conditions carve out the $\mathcal{N} = (0,2)$ moduli space, exactly as the F- and D-term equations do in four dimensions, and both are needed. Without (I.4.51) the pair would be a formal labelling; with it, E_a and J^a are the objects whose vanishing loci are the vacua.

The two $|\cdot|^2$ terms have parallel origins. The $|E_a|^2$ piece is the norm of the Fermi constraint $\bar{D}_+\Lambda_a = E_a$: the bottom component of E_a appears in the Fermi kinetic term as $|E_a(\phi)|^2$, the $\mathcal{N} = (0,2)$ analog of a $4d$ auxiliary field integrating out. The $|J^a|^2$ piece comes from the superpotential (I.4.50): the θ^+ integral and the auxiliary component of Λ_a leave $|J^a(\phi)|^2$, exactly as a $4d$ superpotential leaves $\sum_i |\partial_i W|^2$. So a supersymmetric vacuum needs *both* $E_a = 0$ and $J^a = 0$: they are the two independent contributions to a non-negative potential, and a nonzero value of either breaks supersymmetry.

Conjugating values and exchanging Fermi descriptions. For a Fermi in a complex representation R, E takes values in R and J in R^*. There are two different operations to distinguish:

$$\begin{aligned} &\text{complex conjugation:} \quad (E(\phi), J(\phi))^* = (\bar{E}(\bar{\phi}), \bar{J}(\bar{\phi})), \\ &\text{Fermi replacement:} \quad \Lambda \in R \;\longrightarrow\; \Lambda^\vee \in R^*, \qquad (E^\vee, J^\vee) = (J, E). \end{aligned} \tag{I.4.53}$$

The first line conjugates the chiral fields and produces antiholomorphic functions. The second line keeps the chiral fields fixed and gives holomorphic data for a new Fermi description. The conjugate superfield $\bar{\Lambda}$ obeys a conjugate constraint involving D_+; it is not itself the off-shell superfield satisfying $\bar{D}_+\Lambda^\vee = J$. The replacement conjugates the propagating left-moving fermion and changes the auxiliary-field description. More precisely, order the Yukawa terms as $-\bar{\lambda}_-(\partial_i E)\psi^i_+ + \lambda_-\psi^i_+(\partial_i J) + \text{c.c.}$. Then $\lambda^\vee_- = -\bar{\lambda}_-$, together with $(E^\vee, J^\vee) = (J, E)$, reproduces these terms. The minus sign is a fermion phase in this component ordering, not a relative minus between the two holomorphic functions.

We use the plus exchange in the second line, with the Fermi and auxiliary phases chosen accordingly. No complex conjugates of E or J occur in that line, and no relative minus sign is inserted. A further Fermi rephasing $\Lambda^\vee \mapsto e^{i\alpha}\Lambda^\vee$ gives $E^\vee = e^{i\alpha}J$, $J^\vee = e^{-i\alpha}E$; the phases cancel in $E^\vee J^\vee$. Thus the norm potential and the traced closure condition are unchanged, since $\mathrm{Tr}(JE) = \mathrm{Tr}(EJ)$. This is the exchange meant when the conjugate-representation Fermi is used instead of the original one. Ordinary complex conjugation alone does not perform it.

The closure condition. The two data are not independent as functions. The interaction (I.4.50) is invariant under the single right-moving supercharge only if they satisfy the trace condition

$$\sum_a \mathrm{Tr}\left(E_a\, J^a\right) \;=\; 0, \tag{I.4.54}$$

the sum over all Fermi multiplets, the trace over gauge indices. This is the $2d$ analog of the trace condition that makes a $4d$ superpotential gauge invariant, and it is the

statement that $\bar{D}_+$ acting on the interaction closes,

$$\bar{D}_+(\Lambda_a J^a) \;=\; E_a J^a - \Lambda_a \bar{D}_+ J^a, \tag{I.4.55}$$

so that gauge-invariance plus the chirality of J^a leaves exactly the trace (I.4.54) to cancel. It is not automatic: a set of E and J chosen at random will not satisfy it, and a theory whose data violate it is not a consistent $\mathcal{N} = (0,2)$ theory. In a brick or toric phase each function splits into two signed terms, $E_a = E_a^+ - E_a^-$ and $J^a = J^{a,+} - J^{a,-}$, and (I.4.54) is the statement that the four products cancel in pairs, the plaquette closure of the tiling.

Worked instance: the $\mathbf{C}^4$ brick and its orbifold

The cleanest example where the closure (I.4.54) is a genuine identity, not an accident, is the $\mathbb{C}^4$ brane brick model of Volume VI. There the chiral fields are four matrices $X_1, \ldots, X_4$ (the four coordinates of $\mathbb{C}^4$), and the three Fermi data are commutators

$$E_a \;=\; [\,X_4,\, X_a\,], \qquad J^a \;=\; [\,X_{a+1},\, X_{a+2}\,], \qquad a = 1,2,3, \tag{I.4.56}$$

indices cyclic. The E- and J-index pairs differ for every a, so the two are genuinely distinct functions. The trace sum is

$$\sum_{a=1}^{3} \mathrm{Tr}\big([\,X_4, X_a\,]\,[\,X_{a+1}, X_{a+2}\,]\big) \;=\; 0 \tag{I.4.57}$$

identically, for *any* matrices: it is the Jacobi identity, written as a trace. Expanding one term, $\mathrm{Tr}([X_4,X_1][X_2,X_3])$, and summing the three cyclic images, every monomial $\mathrm{Tr}(X_iX_jX_kX_l)$ that appears is cancelled by another with the opposite sign. The matrices genuinely do not commute, so (I.4.57) is a nontrivial cancellation, not a term-by-term zero. It also singles out the right partner: the wrong pairing, for instance

$$\sum_a \mathrm{Tr}\big(E_a\, E_a\big) \;=\; \sum_a \mathrm{Tr}\big([X_4,X_a]^2\big) \;\neq\; 0, \tag{I.4.58}$$

does not vanish, so the closure is the diagnostic that fixes J^a given E_a.

The identity survives orbifolding. The $\mathbb{C}^4/\mathbb{Z}_4$ diagonal orbifold, weights $(1,1,1,1)$ satisfying the Calabi–Yau condition

$$1+1+1+1 \;=\; 0 \bmod 4, \tag{I.4.59}$$

projects the four X matrices to block form and distributes the multiplets over four gauge nodes. The Volume VI count is

$$\mathbb{C}^4/\mathbb{Z}_4: \quad 4 \text{ nodes}, \quad 16 \text{ chiral}, \quad 12 \text{ Fermi}, \tag{I.4.60}$$

and the projected commutators still obey $\sum_a \mathrm{Tr}(E_a J^a) = 0$, inherited from the parent Jacobi because the projection commutes with the trace. The Calabi–Yau condition is essential: an orbifold whose weights do not sum to zero modulo n has inconsistent

Fermi charges and does not descend to a consistent $\mathcal{N}=(0,2)$ theory. This is the field-theory content Volume VI reads off a brane tiling; the scalar potential (I.4.51), the vacua (I.4.52), and the closure (I.4.54) are its field-theory home.

Common misconception: a $\mathcal{N}=(0,2)$ theory has a single superpotential

A $\mathcal{N}=(0,2)$ theory carries *two* holomorphic data per Fermi multiplet, the pair (E_a, J^a): E_a enters $\bar{D}_+\Lambda_a = E_a$, and J^a enters $W_{\mathcal{N}=(0,2)} = \int d\theta^+ \sum_a \Lambda_a J^a$. They are the two halves of the scalar potential $V \supset \sum_a |E_a|^2 + \sum_a |J^a|^2 + V_D$, so the supersymmetric vacua obey *both* $E_a = 0$ and $J^a = 0$. Supersymmetry of the interaction is not automatic: it requires $\sum_a \mathrm{Tr}(E_a J^a) = 0$, the $2d$ analog of the $4d$ trace condition. Ordinary Hermitian conjugation complex-conjugates function values. Replacing the Fermi by a conjugate-representation Fermi instead exchanges the holomorphic roles, $(E^\vee, J^\vee) = (J, E)$, with the phase convention stated above. A $\mathcal{N}=(2,2)$ theory is the special case in which J and E are fixed by the extended supersymmetry, worked in §I.4.6; genuinely $\mathcal{N}=(0,2)$ theories keep E and J independent.

I.4.6 $\mathcal{N}=(0,2)$ bundles and deformations of $\mathcal{N}=(2,2)$ GLSMs

The most important reason $\mathcal{N}=(0,2)$ theories matter, beyond quiver technology, is geometric. A $\mathcal{N}=(2,2)$ nonlinear sigma model on a Kähler target X has both its fermions valued in the tangent bundle TX, the left-movers locked to the right-movers by extended supersymmetry. A $\mathcal{N}=(0,2)$ model relaxes that locking: the right-movers still couple to TX, but the left-moving Fermi fermions couple to an independent holomorphic vector bundle $E \to X$,

$$\mathcal{N}=(2,2)\ \text{NLSM}:\ \psi_\pm \in TX, \qquad\qquad \mathcal{N}=(0,2)\ \text{NLSM}:\ \psi_+ \in TX,\ \lambda_- \in E, \tag{I.4.61}$$

and the $\mathcal{N}=(2,2)$ theory is recovered as the special case $E = TX$. The freedom to choose $E \neq TX$ is what makes $\mathcal{N}=(0,2)$ models a genuinely larger world (heterotic compactifications, bundle-deformed Calabi–Yau models). But E is not arbitrary: consistency of the quantum theory demands anomaly conditions, the geometric avatar of the $2d$ gauge and gravitational anomalies of the next section,

$$\mathrm{ch}_2(E) \;=\; \mathrm{ch}_2(TX), \qquad c_1(E) \;\equiv\; c_1(TX) \pmod 2. \tag{I.4.62}$$

The first cancels the local sigma-model anomaly between left- and right-moving fermions; the second is the mod-two condition needed to define their fermion determinants globally. They are the target image of the chiral fermion traces that §I.4.7 treats as the $2d$ gauge and gravitational anomalies, so the E/J interactions give a local description of the left-moving bundle.

Ungauged versus gauged: where E comes from. The $\mathcal{N}=(2,2)$ locus $E = TX$ locks the two data by extended supersymmetry, but *how* they lock depends on whether the theory is gauged. For an *ungauged* $\mathcal{N}=(2,2)$ Landau–Ginzburg model with fields

Φ_a and superpotential W, a $\mathcal{N}=(0,2)$ subalgebra splits each chiral into a $\mathcal{N}=(0,2)$ chiral plus a paired Fermi Γ_a, with data

$$E_a \;=\; 0, \qquad J^a \;=\; \frac{\partial W}{\partial \Phi_a} \qquad \text{(ungauged } \mathcal{N}=(2,2) \text{ LG)}, \tag{I.4.63}$$

so E vanishes and J is the gradient of one superpotential. For a $\mathcal{N}=(2,2)$ Wess–Zumino model with Φ_1, Φ_2 and $W=\Phi_1\Phi_2^2$, read as $\mathcal{N}=(0,2)$,

$$J^1 \;=\; \frac{\partial W}{\partial \Phi_1} \;=\; \Phi_2^2, \qquad J^2 \;=\; \frac{\partial W}{\partial \Phi_2} \;=\; 2\,\Phi_1\Phi_2, \qquad E_a = 0, \tag{I.4.64}$$

and the potential reduces to $|\partial_1 W|^2 + |\partial_2 W|^2$, the ordinary F-term potential. This is why one superpotential suffices for an ungauged model.

For a *gauged* $\mathcal{N}=(2,2)$ GLSM the story is different, and this is the point of the subsection: E is *not* zero. The extra ingredient is the $\mathcal{N}=(2,2)$ vector, whose $\mathcal{N}=(0,2)$ decomposition (I.4.10) contains the $\mathcal{N}=(0,2)$ field strength Σ. A $\mathcal{N}=(2,2)$ chiral Φ_i of gauge charge Q_i splits into a $\mathcal{N}=(0,2)$ chiral and a Fermi Γ_i whose E-term is fixed by the gauge coupling,

$$\bar{D}_+\Gamma_i \;=\; E_i \;=\; \sqrt{2}\,Q_i\,\Sigma\,\Phi_i, \tag{I.4.65}$$

a nonzero E built from the field strength and the charged field. This is the $\mathcal{N}=(0,2)$ statement of the $2d$ twisted mass $Q_i\Sigma$ of §I.4.3: the gaugino partner of that mass is exactly the E-term (I.4.65). So a gauged $\mathcal{N}=(2,2)$ GLSM, read as $\mathcal{N}=(0,2)$, has $E_i \neq 0$ from the gauge coupling and J^{Γ_i} from the superpotential, and the two data must together satisfy the closure (I.4.54).

Worked instance: the gauged quintic E-term and its closure by Euler's theorem

The quintic gauged linear sigma model of §I.4.3 makes this concrete, and the closure is a deep, one-line calculation. The fields are five ϕ_i of charge $Q_i=+1$ and one P of charge $Q_P=-5$, with $W=P\,G_5(\phi)$. Reading it as $\mathcal{N}=(0,2)$, each $\mathcal{N}=(2,2)$ chiral becomes a $\mathcal{N}=(0,2)$ chiral plus a Fermi, and the two data are the gauged E-terms (I.4.65) together with the superpotential J's,

$$\begin{aligned} E_i \;&=\; \sqrt{2}\,\Sigma\,\phi_i, & E_P \;&=\; -5\sqrt{2}\,\Sigma\,P, \\ J^i \;&=\; P\,\frac{\partial G_5}{\partial \phi_i}, & J^P \;&=\; G_5(\phi), \end{aligned} \tag{I.4.66}$$

the E's carrying the charges $+1$ and -5 through (I.4.65), the J's the derivatives of $W=P\,G_5$. This is the correction the earlier ungauged reading misses: for the gauged theory $E\neq 0$. Now form the closure sum (I.4.54),

$$\sum_i E_iJ^i + E_PJ^P \;=\; \sqrt{2}\,\Sigma\,P\sum_i \phi_i\,\frac{\partial G_5}{\partial \phi_i} - 5\sqrt{2}\,\Sigma\,P\,G_5 \;=\; \sqrt{2}\,\Sigma\,P\Big(\sum_i \phi_i\,\partial_i G_5 - 5\,G_5\Big), \tag{I.4.67}$$

the common factor $\sqrt{2}\,\Sigma P$ pulled out. The bracket vanishes by Euler's theorem for a homogeneous function: G_5 is homogeneous of degree five, so

$$\sum_i \phi_i \frac{\partial G_5}{\partial \phi_i} \;=\; 5\,G_5 \quad\Longrightarrow\quad \sum_i E_i J^i + E_P J^P \;=\; 0. \tag{I.4.68}$$

The closure follows from gauge invariance of the superpotential, equivalently the quasi-homogeneity identity $\sum_i Q_i \Phi_i \partial_i W = 0$. For the quintic the same integer 5 appears in three places: gauge invariance of $W = PG_5$ (the charge $Q_P = -5$ balancing the degree $\deg G_5 = 5$), the Euler homogeneity $\sum_i \phi_i \partial_i G_5 = 5G_5$, and the Calabi–Yau charge sum $5(+1) + (-5) = 0$. A superpotential of the wrong degree is not gauge invariant, and then its E/J do not close: with $Q_P = -5$ fixed but $\deg G_d = d$, the same computation leaves

$$\sum_i E_i J^i + E_P J^P \;=\; \sqrt{2}\,\Sigma\, P\,(d-5)\,G_d \;\neq\; 0 \qquad (d \neq 5), \tag{I.4.69}$$

so closure requires $d = 5 = -Q_P$, which is precisely gauge invariance of W. In this hypersurface model that same integer 5 also enters the Calabi–Yau charge assignment, but the two statements are distinct: E/J closure is the supersymmetry / gauge-invariance condition, while the Calabi–Yau condition is the separate statement that the axial R-symmetry is non-anomalous.

Worked instance: a $\mathcal{N} = (0,2)$ deformation of the quintic GLSM

The freedom to choose $E \neq TX$ is best seen as a deformation of the $\mathcal{N} = (2,2)$ point just worked. At the $\mathcal{N} = (2,2)$ (tangent-bundle) locus the Fermi data are the locked pair (I.4.66). A $\mathcal{N} = (0,2)$ deformation replaces the locked J^i by more general holomorphic functions $J^i = J^i(\phi, P)$ and deforms the E_i away from (I.4.65), subject to two constraints: the closure $\sum_a \mathrm{Tr}(E_a J^a) = 0$ of (I.4.54) (supersymmetry) and the bundle-anomaly conditions (I.4.62). The deformed model describes the same Calabi–Yau X but with the left-moving fermions coupled to a bundle $E \neq TX$: a *bundle-deformed* gauged linear sigma model. The anomaly conditions bound the family, keeping $\mathrm{ch}_2(E) = \mathrm{ch}_2(TX)$ and $c_1(E) \equiv c_1(TX) \bmod 2$, so the surviving deformations are those preserving the two characteristic classes.

The lesson is the shape of the moduli space. The $\mathcal{N} = (2,2)$ locus (complex-structure and Kähler deformations) sits inside a larger $\mathcal{N} = (0,2)$ moduli space (bundle deformations with $E \neq TX$), the enlargement the extra freedom in E and J that the anomaly conditions do not kill. So $\mathcal{N} = (0,2)$ models are the field-theory description of heterotic compactifications on (X, E), their marginal deformations the bundle moduli. Volume VI reads this data off a brane tiling, and the home of "which deformations survive" is (I.4.62) together with the closure (I.4.54).

I.4.7 2d anomalies and c-extremization

Chirality makes two dimensions dangerous, and the danger is quantified by chiral anomalies of two kinds, handled in opposite ways. It is also chirality that lets an

extremization principle fix the exact superconformal R-symmetry and the central charges. We take the anomalies first, then c-extremization, which is built from them.

The gauge anomaly. A $2d$ chiral fermion in a gauge representation contributes to the gauge two-point anomaly with a sign set by its worldsheet chirality: right-movers one sign, left-movers the other. Collecting the $\mathcal{N}=(0,2)$ content, the right-moving fermions in chiral multiplets enter with a plus, and the left-moving fermions in Fermi multiplets and the gaugino (an adjoint left-mover) enter with a minus. The theory is consistent only if the total vanishes,

$$\mathrm{Tr}_{\mathrm{chiral}} T^A T^B \;-\; \mathrm{Tr}_{\mathrm{Fermi}} T^A T^B \;-\; \mathrm{Tr}_{\mathrm{adj}} T^A T^B \;=\; 0. \tag{I.4.70}$$

On the $\mathbb{C}^4$ brick, where the content per node is four adjoint chirals, three adjoint Fermis, and one adjoint gaugino, this reads in units of the adjoint index

$$4\,T(\mathrm{adj}) \;-\; 3\,T(\mathrm{adj}) \;-\; T(\mathrm{adj}) \;=\; 0, \tag{I.4.71}$$

the clean $4-3-1=0$. The gaugino term is load-bearing: dropping it leaves $4-3=1\neq 0$, an inconsistent theory. For a $\mathcal{N}=(2,2)$ theory the cancellation is automatic, and (I.4.9) is why: each $\mathcal{N}=(2,2)$ chiral supplies one $\mathcal{N}=(0,2)$ chiral and one $\mathcal{N}=(0,2)$ Fermi in the same representation, so their traces cancel. For n $\mathcal{N}=(2,2)$ chirals plus one $\mathcal{N}=(2,2)$ vector, the $\mathcal{N}=(0,2)$ counts are $n+1$ chirals (the n matter chirals plus the vector's adjoint chiral), n Fermis, and one vector, so (I.4.70) reads

$$(n+1) \;-\; n \;-\; 1 \;=\; 0 \tag{I.4.72}$$

identically. The Gadde–Gukov–Putrov node of §I.4.8 is the worked nontrivial case: a $U(N_c)$ gauge group with N_1 fundamental chirals, N_2 antifundamental chirals, and N_3 Fermis is free of the $SU(N_c)$ gauge anomaly, for $N_c\geq 2$, precisely when

$$N_1+N_2-N_3-2N_c \;=\; 0, \tag{I.4.73}$$

fixing the balanced rank $N_c=\frac{1}{2}(N_1+N_2-N_3)$, the $2N_c$ the gaugino's adjoint contribution. At $(N_1,N_2,N_3)=(5,3,4)$ the balanced rank is $N_c=2$, and $5+3-4-4=0$. Reading the coefficient block by block makes the sign structure explicit: each fundamental chiral contributes $+T(\square)=+\frac{1}{2}$ (right-moving), each Fermi $-T(\square)=-\frac{1}{2}$ (left-moving), and the gaugino $-T(\mathrm{adj})=-N_c$ (left-moving adjoint), so the total is

$$\tfrac{1}{2}(N_1+N_2) \;-\; \tfrac{1}{2}N_3 \;-\; N_c \;=\; \tfrac{1}{2}\big(N_1+N_2-N_3-2N_c\big), \tag{I.4.74}$$

zero exactly on the balanced node for $N_c\geq 2$. The full $U(N_c)$ model also contains N_1N_2 gauge-singlet Fermis and two determinant Fermis, specified below; the latter cancel the diagonal abelian anomaly. There is no $SU(1)$ anomaly at rank one. In this family the rank N_c is not free: it is *determined* by requiring the gauge anomaly to vanish, which is why $\mathcal{N}=(0,2)$ SQCD has a fixed gauge rank for given flavor content, and it is this same balancing that the triality of §I.4.8 preserves as it permutes the flavor blocks.

The gravitational anomaly. The gravitational anomaly is the net worldsheet chirality of the massless fermions,

$$c_R - c_L \;=\; \mathrm{Tr}\,\gamma^3, \tag{I.4.75}$$

where $\gamma^3 = +1$ on a right-mover and -1 on a left-mover, so $\mathrm{Tr}\,\gamma^3$ counts right-movers minus left-movers. Unlike the gauge anomaly, this is generically *nonzero*, and a nonzero value is allowed and physical. It is a 't Hooft anomaly of the $2d$ Lorentz (chirality) symmetry, invariant along the renormalization-group flow, and it must *match* between two descriptions of the same theory rather than cancel. A content of three right-movers and one left-mover has $c_R - c_L = 3 - 1 = 2$, a consistent chiral theory. The $\mathbb{C}^4/\mathbb{Z}_4$ brick, with 16 right-moving chiral fermions, 12 left-moving Fermi fermions, and 4 left-moving gauginos, has

$$c_R - c_L \;=\; 16 - 12 - 4 \;=\; 0, \tag{I.4.76}$$

a non-chiral theory. The matching statement is the working tool: two descriptions of the same infrared physics must report the same $c_R - c_L$, one of the invariants the three triality frames share. As a matching example, two spectra grouped differently but sharing the net chirality,

$$\begin{aligned} &\text{A:}\quad 4 \text{ right} + 2 \text{ left}: \quad c_R - c_L = 4 - 2 = 2, \\ &\text{B:}\quad (3+1) \text{ right} + 2 \text{ left}: \quad c_R - c_L = (3+1) - 2 = 2, \end{aligned} \tag{I.4.77}$$

report the same $c_R - c_L = 2$: the anomaly sees only the net right-minus-left count, and being RG-invariant it is the same in the ultraviolet and infrared. This is what makes it a diagnostic: if two ultraviolet Lagrangians claiming the same infrared theory report different $c_R - c_L$, one is wrong, and the computation is a one-line count of fermions weighted by γ^3.

Common misconception: the gravitational anomaly must cancel

Two dimensions has two distinct chiral-anomaly conditions, handled differently. The *gauge* anomaly $\mathrm{Tr}_{\mathrm{chiral}} - \mathrm{Tr}_{\mathrm{Fermi}} - \mathrm{Tr}_{\mathrm{adj}} = 0$ must cancel for the theory to exist. The *gravitational* anomaly $c_R - c_L = \mathrm{Tr}\,\gamma^3$, the net fermion chirality, is generically nonzero; it is a 't Hooft anomaly of the $2d$ Lorentz symmetry, RG-invariant, and it must *match* between descriptions (for instance the three triality frames), not cancel. A nonzero $c_R - c_L$ is allowed and physical; an uncancelled *gauge* anomaly is fatal.

c-extremization. The exact right-moving R-symmetry of a $2d$ $\mathcal{N} = (0,2)$ (or $\mathcal{N} = (2,2)$) fixed point is fixed by an extremization principle, the two-dimensional member of the extremization through-line whose $4d$ prototype is the a-maximization of Chapter 3. The right-moving central charge of the fixed point is the 't Hooft anomaly of its right-moving R-symmetry, so for a trial right-moving R the trial central charge is

$$c_R^{\mathrm{tr}}(R) \;=\; 3\,\mathrm{Tr}\left[\gamma^3\, R^2\right], \tag{I.4.78}$$

a sum over fermions weighted by the chirality $\gamma^3 = \pm 1$ and the square of the R-charge, the $2d$ analog of the trace that builds a in four dimensions. A trial R ranges over the exact R mixed with the abelian flavors, $R = R_0 + \sum_F s_F\, F$, and the exact superconformal R is the stationary point, where the mixed anomaly with each flavor vanishes,

$$k_{RF} \;=\; \mathrm{Tr}\left[\gamma^3\, R\, F\right] \;=\; 0, \tag{I.4.79}$$

one equation per flavor. Solving (I.4.79) gives R_{exact}, evaluating (I.4.78) there gives c_R, and the left-moving central charge follows from the gravitational anomaly,

$$c_L \;=\; c_R - \operatorname{Tr}\gamma^3. \tag{I.4.80}$$

The hypotheses match a-maximization: the theory must flow to a good candidate infrared fixed point, the fixed point must be unitary with a normalizable vacuum and Virasoro energies bounded below, the trial R must be a genuine symmetry, and all abelian mixing currents must be included. Using ultraviolet anomalies also requires that no unaccounted accidental infrared symmetry mixes into R. A formal fermion list violating any of these can be fed through (I.4.78), but its output is not the central charge of a unitary theory.

Stationary, not maximal. Here is the sharp difference from a-maximization. In four dimensions the exact R *maximizes* a, and the Hessian is negative definite. In two dimensions the exact R only *extremizes* c_R^{tr}, and the stationary point is generically a *saddle*. Differentiating (I.4.78) twice in the mixing parameters, with $\partial R/\partial s_F = F$, gives the γ^3-weighted flavor metric

$$\frac{\partial^2 c_R^{\text{tr}}}{\partial s_F\,\partial s_{F'}} \;=\; 6\operatorname{Tr}\left[\gamma^3\,F\,F'\right], \tag{I.4.81}$$

and because γ^3 takes both signs this metric *can be indefinite*. To see it on a two-flavor example, take one right-moving chiral ($\gamma^3 = +1$) carrying $F_1 = 1$, $F_2 = 0$ and one left-moving Fermi ($\gamma^3 = -1$) carrying $F_1 = 0$, $F_2 = 1$. Then $c_R^{\text{tr}}(s_1, s_2) = 3[(+1)(R_0 + s_1)^2 + (-1)(R_0 + s_2)^2]$, and its Hessian is

$$\frac{\partial^2 c_R^{\text{tr}}}{\partial s_a\,\partial s_b} \;=\; 6\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{eigenvalues } \{+6,\ -6\}, \tag{I.4.82}$$

one positive and one negative, a genuine saddle: the trial central charge increases along the right-moving flavor and decreases along the left-moving one, so no stationary point is a local maximum. What c-extremization fixes is the stationarity condition and the extremum type, not a maximum.

The saddle-versus-maximum split is structural, not accidental. Four-dimensional a-maximization extremizes a *cubic* anomaly and has a negative-definite Hessian at the physical fixed point. Two-dimensional c-extremization extremizes a *quadratic* anomaly $\operatorname{Tr}[\gamma^3 R^2]$ whose Hessian (I.4.81) is weighted by γ^3. Right-moving flavor currents give positive curvature and left-moving currents negative curvature; when both are present the stationary point is a saddle. The same chirality that made two dimensions dangerous turns the maximization of four dimensions into an extremization here, and one must *solve* $k_{RF} = 0$ rather than search for the largest c_R.

Worked instance: the $\mathcal{N} = (0,2)$ A_{k+1} minimal model (positive c_R)

The physical worked example is a genuine unitary $\mathcal{N} = (0,2)$ superconformal field theory: the $\mathcal{N} = (0,2)$ description of the A_{k+1} Landau–Ginzburg minimal model. Take

one $\mathcal{N}=(0,2)$ chiral Φ (with right-moving fermion ψ_+, $\gamma^3=+1$) and one $\mathcal{N}=(0,2)$ Fermi Λ (with left-moving λ_-, $\gamma^3=-1$), coupled by

$$J \;=\; \Phi^{k+1}, \qquad E=0. \tag{I.4.83}$$

The superpotential $\int d\theta^+\,\Lambda\,\Phi^{k+1}$ is the $\mathcal{N}=(0,2)$ face of the $\mathcal{N}=(2,2)$ Landau–Ginzburg model with $W=\Phi^{k+2}$. Viewed with only $\mathcal{N}=(0,2)$ supersymmetry, the constraint $R(\Lambda)+(k+1)R(\phi)=1$ still permits one trial mixing. Writing $r=R(\phi)$ gives $R(\psi_+)=r-1$ and $R(\lambda_-)=1-(k+1)r$. The anomaly trace is then $c_R^{\mathrm{tr}}(r)=3[(r-1)^2-(1-(k+1)r)^2]$; its derivative vanishes at $r=1/(k+2)$ for $k>0$. This reproduces the quasi-homogeneous $\mathcal{N}=(2,2)$ assignment,

$$R(\phi) \;=\; \frac{1}{k+2}, \qquad R(\psi_+) \;=\; R(\phi)-1 \;=\; -\frac{k+1}{k+2}, \qquad R(\lambda_-) \;=\; \frac{1}{k+2}, \tag{I.4.84}$$

the last from $R(J)=(k+1)/(k+2)$. Evaluating the trial central charge (I.4.78) on these two fermions,

$$c_R \;=\; 3\Big[\,(+1)\Big(\tfrac{k+1}{k+2}\Big)^2 \;+\; (-1)\Big(\tfrac{1}{k+2}\Big)^2\,\Big] \;=\; 3\cdot\frac{(k+1)^2-1}{(k+2)^2} \;=\; \frac{3k}{k+2}, \tag{I.4.85}$$

positive for every $k\geq 1$: a genuine unitary superconformal field theory. It equals $3\hat{c}$ with the Landau–Ginzburg central charge $\hat{c}=k/(k+2)=1-2R(\phi)$, matching (I.4.18). The smallest nontrivial case is $k=2$,

$$k=2: \quad R(\phi)=\tfrac{1}{4}, \quad c_R=\frac{3\cdot 2}{2+2}=\frac{3}{2}, \tag{I.4.86}$$

the $c_R=3/2$ of the $k=2$ member of the $\mathcal{N}=2$ minimal-model series. As k grows, $c_R=3k/(k+2)\to 3$ from below. Here the stationary anomaly trace selects the physical R-charge and yields a positive central charge.

Worked instance: a formal saddle example (not a unitary SCFT)

To exhibit the indefinite Hessian directly one needs a free mixing parameter, and a minimal example is a *formal* fermion list chosen only to expose the saddle, not a unitary theory. Take two right-moving chiral fermions ($\gamma^3=+1$) with flavor charges $F=+1$ and $F=+2$, and two left-moving Fermi fermions ($\gamma^3=-1$) with $F=-1$ and $F=+1$. The γ^3-weighted flavor norm is

$$\mathrm{Tr}\,\big[\gamma^3\,F^2\big] \;=\; (1+4)-(1+1) \;=\; 3 \;\neq\; 0, \tag{I.4.87}$$

so the mixing genuinely moves the mixed anomaly. Writing $R(s)=R_0+s\,F$ with a non-extremal seed $R_0=0.2$, the stationarity $k_{RF}(s)=\mathrm{Tr}[\gamma^3 RF]=0$ is one linear equation,

$$k_{RF}(s) \;=\; 0.6+3s \;=\; 0 \quad\Longrightarrow\quad s_* \;=\; -0.2, \tag{I.4.88}$$

with the constant $0.6=k_{RF}(0)$ the anomaly at the seed and the slope 3 the flavor norm (I.4.87). At s_* the four R-charges are $R=0,\,-0.2,\,0.4,\,0$, and the trial central charge there is

$$c_R \;=\; 3\big[(0)^2+(0.2)^2-(0.4)^2-(0)^2\big] \;=\; 3(0.04-0.16) \;=\; -0.36. \tag{I.4.89}$$

The negative output is the point: a negative "central charge" signals a fermion list that is *not* a unitary superconformal field theory. The extremization algebra runs on any anomaly data, but only a genuine candidate fixed point (like the minimal model above) returns a physical c_R. This example displays the algebraic stationary point without assigning it a physical central charge.

Worked instance: the symmetric brick (R forced by structure)

The contrast partner is the case where symmetry does the work, the $2d$ analog of the symmetric-R quiver of Chapter 3. For the $\mathbb{C}^4/\mathbb{Z}_4$ $\mathcal{N}=(0,2)$ brick the R-charge is forced by the E/J and superpotential structure with no free mixing. A Fermi obeys $\bar{D}_+\Lambda_a = E_a$ with E_a bilinear in the chirals, so $R(E_a) = 2R_X$ with R_X the common chiral R-charge; the $\bar{D}_+$ carries R-charge $+1$, so $R(\Lambda_a) = R(E_a) - 1 = 2R_X - 1$. The superpotential $W_{\mathcal{N}=(0,2)} = \int d\theta^+ \sum_a \Lambda_a J^a$ must carry R-charge one, and J^a is also bilinear, $R(J^a) = 2R_X$, so

$$R(\Lambda_a) + R(J^a) \;=\; (2R_X - 1) + 2R_X \;=\; 1 \quad\Longrightarrow\quad R_X \;=\; \tfrac{1}{2}, \quad R(\Lambda_a) = 0. \tag{I.4.90}$$

Within the common-R sector these equations force $R_X = 1/2$ and $R(\Lambda_a) = 0$. There is no remaining common-R direction to extremize. Likewise, $3R = 2$ fixes the common charge in the $4d$ dP_0 quiver before any maximization. These are structural charge constraints. An additional extremization test would require an allowed flavor-mixing direction outside this restricted sector and its anomaly trace in (I.4.79).

I.4.8 $\mathcal{N}=(0,2)$ triality

Three different $2d$ $\mathcal{N}=(0,2)$ gauge theories can flow to the same infrared $\mathcal{N}=(0,2)$ superconformal field theory. This is the two-dimensional cousin of Seiberg duality, but with a decisive difference: it is an *order-three* relation, a triality, not an order-two involution. For the $\mathcal{N}=(0,2)$ brick this is the order-three move in the graded-quiver family. We pin the Gadde–Gukov–Putrov conventions exactly, because the flavor labels are convention-sensitive and the field-theory content is more than a rank map.

The complete Gadde–Gukov–Putrov node. Keep this chapter's flavor labeling: N_1 counts fundamental chirals P, N_2 antifundamental chirals Φ, and N_3 fundamental Fermis Ψ. Relative to the original SQCD tables, $N_b = N_2$, $N_f = N_3$ and $2N_c + N_f - N_b = N_1$; we conjugate the gauge and flavor representation conventions. The two uncoupled determinant Fermis can then be replaced by their conjugate-representation Fermis, so we take both in det, of diagonal charge $+N_c$. With a fundamental of charge $+1$, the complete matter is

field	multiplet	$U(N_c)$	$SU(N_1)\times SU(N_2)\times SU(N_3)$	q_{diag}
P	chiral	$\square$	$(\overline{\square},1,1)$	$+1$
Φ	chiral	$\overline{\square}$	$(1,\square,1)$	-1
Ψ	Fermi	$\square$	$(1,1,\overline{\square})$	$+1$
Γ	Fermi	1	$(\square,\overline{\square},1)$	0
$\Omega_{1,2}$	Fermi	det	$(1,1,1)$	$+N_c$

(I.4.91)

There are N_1N_2 singlet Fermis Γ, distinct from the N_3 charged Fermis Ψ, and a doublet $\Omega_{1,2}$ under an additional $SU(2)$ flavor group. The vector contains one left-moving adjoint gaugino. All E functions vanish in this SQCD presentation. The only nonzero J is $J_\Gamma = \Phi P$, with gauge and flavor contractions: $\int d\theta^+ \, \Gamma\Phi P + \text{c.c.}$ The J functions of Ψ and $\Omega_{1,2}$ vanish. The coupling is gauge invariant, linear in a Fermi, and satisfies $\sum EJ = 0$.

For $N_c \geq 2$, the nonabelian condition is

$$N_1 + N_2 - N_3 - 2N_c \;=\; 0 \quad\Longrightarrow\quad N_c \;=\; \tfrac{1}{2}(N_1 + N_2 - N_3), \tag{I.4.92}$$

where $T(\square) = 1/2$ and $T(\text{adj}) = N_c$; neither Γ nor Ω contributes. The diagonal $U(1)$ has a separate coefficient. Each charged fundamental multiplet has N_c components, while each determinant Fermi has one component of charge N_c:

$$\mathcal{A}_{U(1)^2} = N_c(N_1 + N_2 - N_3) - 2N_c^2 = 0. \tag{I.4.93}$$

The adjoint gaugino and Γ have zero diagonal charge. At $N_c = 1$ this abelian equation still applies, but there is no nonabelian factor and hence no $SU(1)$ anomaly calculation. These are local gauge-anomaly tests of the stated representations.

The proposed triality changes the rank by

$$N_c \;\longmapsto\; N_c' \;=\; N_2 - N_c \;=\; \tfrac{1}{2}(N_2 + N_3 - N_1), \tag{I.4.94}$$

and cycles the multiplicities,

$$(N_1, N_2, N_3) \;\longmapsto\; (N_2, N_3, N_1) \tag{I.4.95}$$

Thus the old Φ flavor becomes the P' flavor, the old Ψ flavor becomes the Φ' flavor, and the old P flavor becomes the Ψ' flavor. The same complete table applies with primed ranks and these cyclically relabeled flavor groups. With N_2 artificially held fixed, the rank formula alone is an involution; the combined operation has order three.

Mesons and the fields that remain. A chiral–Fermi bilinear $\Phi\Psi$ gives a Fermi meson Γ' with N_2N_3 components, in $(\square_{N_2}, \overline{\square}_{N_3})$. It is a gauge singlet in the new frame. Its coupling is $J_{\Gamma'} = \Phi' P'$, so $\int d\theta^+ \, \Gamma'\Phi' P' + \text{c.c.}$ is fermionic as required. A chiral–chiral meson $M = \Phi P$ also appears in the intermediate transformation. The original coupling becomes the mass term $\int d\theta^+ \, \Gamma M + \text{c.c.}$, with $J_\Gamma = M$. Its N_1N_2 chiral–Fermi pairs are integrated out. Consequently M and the old Γ are absent from the surviving dual SQCD table. There are no additional singlet-chiral flip fields in this model. The surviving dual has $E = 0$, $J_{\Gamma'} = \Phi' P'$, and $J_{\Psi'} = J_{\Omega'} = 0$.

For the first step this gives

$$\begin{aligned} &U(2) \longrightarrow U(1), \qquad (N_1', N_2', N_3') = (3, 4, 5), \\ &N_{\Gamma'} = N_2N_3 = 12, \qquad N_{(M,\Gamma)\,\text{removed}} = N_1N_2 = 15. \end{aligned} \tag{I.4.96}$$

The surviving $U(1)$ fields are three charge-$+1$ chirals P', four charge--1 chirals Φ', five charge-$+1$ Fermis Ψ', twelve neutral Fermis Γ', and two charge-$+1$ determinant Fermis Ω'. The singlet J couplings contract the three P' and four Φ' flavors. Repeating the

same construction gives a $U(3)$ frame with four P'', five Φ'', three Ψ'', twenty singlet Fermis, and two determinant Fermis of charge $+3$, then the initial $U(2)$ table with fifteen singlet Fermis. This follows the full matter transformation as well as its rank arithmetic.

The order-three return, checked in all three frames. Apply the combined move to an explicit asymmetric balanced triple, $(N_1, N_2, N_3) = (5, 3, 4)$, with $N_c = \frac{1}{2}(5+3-4) = 2$. Recomputing the rank from (I.4.92) after each flavor permutation,

$$\begin{aligned}
&(5,3,4) : N_c = \tfrac{1}{2}(5+3-4) = 2, \quad 5+3-4-2\cdot 2 = 0,\\
&(3,4,5) : N_c = \tfrac{1}{2}(3+4-5) = 1, \quad 3+4-5-2\cdot 1 = 0,\\
&(4,5,3) : N_c = \tfrac{1}{2}(4+5-3) = 3, \quad 4+5-3-2\cdot 3 = 0,
\end{aligned} \tag{I.4.97}$$

these are three evaluations of the balanced-rank polynomial. The second is not an $SU(1)$ anomaly. Using the actual determinant charges, the abelian coefficients in the three frames are

$$U(2) :\ 2(5+3-4) - 2(2)^2 = 8-8 = 0, \qquad U(1) :\ 3+4-5-2(1)^2 = 2-2 = 0,$$

$$U(3) :\ 3(4+5-3) - 2(3)^2 = 18-18 = 0.$$

Omitting the two Ω fields instead leaves $8, 2, 18$, respectively. Tracking $(N_1, N_2, N_3; N_c)$ through the three steps,

$$(5,3,4;\,2) \ \to\ (3,4,5;\,1) \ \to\ (4,5,3;\,3) \ \to\ (5,3,4;\,2), \tag{I.4.98}$$

the intermediate ranks $2, 1, 3$ genuinely different and the third application returning the original data, $T^3 = \mathbb{1}$. The orbit has length three, not two. Without the flavor permutation the bare rank recursion would close at two; the permutation is what makes it a triality. When the flavor numbers coincide the node is self-dual,

$$N_1 = N_2 = N_3 \implies N_2 = 2N_c, \quad N_c' = N_2 - N_c = N_c, \tag{I.4.99}$$

the permutation the identity and the rank fixed: a triality fixed point of order one, a single self-dual frame.

The evidence. The infrared triality is a physical proposal supported by the source's flavor-anomaly, central-charge and equivariant-index comparisons. The rank orbit alone does not establish it. With the complete matter table we can also compute the gravitational coefficient:

$$\operatorname{Tr}\gamma^3 = N_c(N_1 + N_2 - N_3) - N_1N_2 - 2 - N_c^2.$$

On the three frames this is $8-15-2-4 = -13$, $2-12-2-1 = -13$, and $18-20-2-9 = -13$. Here the singlet Fermis, determinant pair and entire $U(N_c)$ gaugino dimension all matter. This is one matched gravitational coefficient, distinct from the gauge-anomaly cancellations. The source also compares the central charges and the genus of §I.4.9. The elliptic genus is the sharpest: a whole function of the fugacities, its equality compares a signed protected counting function; cancellations can

hide states. As a cheap necessary check one can also form a permutation-symmetric combination of the flavor numbers, the toy quantity

$$I \;=\; N_1N_2 + N_2N_3 + N_3N_1 - \tfrac{1}{2}\big(N_1^2 + N_2^2 + N_3^2\big), \tag{I.4.100}$$

symmetric in (N_1, N_2, N_3) and so unchanged by the permutation (I.4.95). This is a toy symmetric check, not a named central-charge or anomaly formula: it is invariant precisely because it is a symmetric function of the flavor multiset, so it gives only a necessary check. On the orbit the three frames each give

$$I \;=\; (15 + 12 + 20) - \tfrac{1}{2}(25 + 9 + 16) \;=\; 47 - 25 \;=\; 22, \tag{I.4.101}$$

the same value (the same multiset reordered), while a mismatched $(6, 3, 4)$ gives $I = 54 - 30.5 = 23.5 \neq 22$ and is not a frame. The check does real work only in the negative direction (rejecting a wrong-multiset candidate); the genuine relation is the anomaly, central-charge, and elliptic-genus matches, not a symmetric polynomial in the ranks. The displayed gravitational match uses the complete surviving matter. The source's central-charge and index comparisons provide further evidence; none of these necessary tests is a proof of infrared equivalence. The dynamical triality discussion is deferred to Volume IV.

frame	N_1	N_2	N_3	N_c
1	5	3	4	2
2 (dualize)	3	4	5	1
3 (dualize)	4	5	3	3
1 (return)	5	3	4	2

Table I.4.2: One $\mathcal{N} = (0, 2)$ triality orbit in Gadde–Gukov–Putrov conventions. Each step applies the rank map $N_c \mapsto N_2 - N_c$ together with the flavor cyclic permutation $(N_1, N_2, N_3) \mapsto (N_2, N_3, N_1)$, and generates the dual matter (permuted charged blocks, surviving singlet Fermis and determinant Fermis). The intermediate ranks $2, 1, 3$ differ and the third step returns the original data ($T^3 = \mathbb{1}$): the orbit has length three, not two.

Common misconception: $\mathcal{N} = (0, 2)$ triality is a $2d$ Seiberg duality

Seiberg duality in four dimensions is an order-two relation: dualizing twice returns the original theory. $\mathcal{N} = (0, 2)$ triality is order *three*. The rank map $N_c \mapsto N_2 - N_c$ alone is an involution, but the triality is the rank map *together with* the flavor cyclic permutation $(N_1, N_2, N_3) \mapsto (N_2, N_3, N_1)$, and the combined move closes only after three applications, $T^3 = \mathbb{1}$. The three frames are three *different* ultraviolet gauge theories, with dual matter (permuted charged blocks, singlet Fermis and determinant Fermis), that flow to the same infrared physics, and the relation is checked by the anomaly, central-charge, and elliptic-genus matches across all three.

Part III. The shared protected interface

The last two sections are the genuinely shared technology. The elliptic genus (§I.4.9) is a $\mathcal{N} = (0, 2)$ object that specializes to $\mathcal{N} = (2, 2)$ and is the sharpest protected check for both worlds, including the triality of Part II and the GLSM phases of Part I. A final comparison table (§I.4.10) reads the two worlds side by side and routes each structure to the construction volume that consumes it.

I.4.9 The elliptic genus

The sharpest protected observable of a $2d$ $\mathcal{N} = (0, 2)$ theory is its elliptic genus, the supersymmetric partition function on a two-torus. It is the two-dimensional analog of the $4d$ superconformal index of Chapter 3: a flavored, signed counting function, invariant under continuous deformations, that matches across dualities. It is a $\mathcal{N} = (0, 2)$ object that specializes to $\mathcal{N} = (2, 2)$, which is why it is the shared tool that closes the chapter rather than a Part-I or Part-II section.

A protected index. For a general $\mathcal{N} = (0, 2)$ theory, the torus trace in the Ramond–Ramond sector weights $(-1)^F$ by the left- and right-moving Hamiltonians and by flavor charges K_a. In a $\mathcal{N} = (2, 2)$ theory with a left-moving R charge J_L, one may also turn on its fugacity y. Keeping these two gradings distinct gives

$$\begin{aligned} Z^{(0,2)}_{T^2}(\tau, \boldsymbol{x}) &= \mathrm{Tr}_{RR}\left[(-1)^F q^{H_L} \bar{q}^{H_R} \prod_a x_a^{K_a}\right], \\ Z^{(2,2)}_{T^2}(\tau, y, \boldsymbol{x}) &= \mathrm{Tr}_{RR}\left[(-1)^F q^{H_L} \bar{q}^{H_R} y^{J_L} \prod_a x_a^{K_a}\right], \end{aligned} \qquad q = e^{2\pi i \tau}, \quad y = e^{2\pi i z}. \tag{I.4.102}$$

For a compact theory with a discrete spectrum and a convergent supersymmetric trace, positive-H_R states pair and cancel. This gives holomorphy in τ. Deformation invariance also assumes that states do not escape to a continuum. A weak Jacobi form further requires holomorphy in the elliptic variables, the appropriate integral charge lattice (or specified multiplier), and the weak lower bound on Fourier powers in the chosen RR normalization. Under these hypotheses the anomaly data fix the modular and elliptic transformation laws.

Flavor fugacities can regulate zero modes when noncompact directions are present. Such an equivariant answer may instead be meromorphic in those fugacities. One must specify the supersymmetric regulator and keep fugacities away from poles; removing it need not leave a finite holomorphic genus. A continuous spectrum can contribute a nonholomorphic completion through unequal boson and fermion densities. Gauge-anomaly cancellation alone excludes none of these possibilities.

Place it among the indices already known. The Witten index $\mathrm{Tr}(-1)^F$ counts ground states with signs, one number. The elliptic genus refines it: it keeps the dependence

on τ (through q^{H_L}) and on the fugacities, so it counts the tower of right-moving-supersymmetric states graded by left-moving energy. For a general $\mathcal{N}=(0,2)$ theory, define $I_0(\boldsymbol{x}) = [q^0]Z^{(0,2)}_{T^2}$ as the coefficient of the unshifted q^0 term. It can vanish even when the leading ground-state contribution is nonzero: the free Fermi factor begins with $q^{1/12}(x^{1/2}-x^{-1/2})$. Reading a ground-state index then requires removing that specified leading power. The raw coefficient need not equal a $q\to 0$ limit or a genus of the target alone. A more specific geometric limit holds for a compact $\mathcal{N}=(2,2)$ Calabi–Yau sigma model on a complex d-fold X with its left R fugacity y retained and all flavor fugacities set to one,

$$\begin{aligned} [q^0]Z^{(0,2)}_{T^2} &= I_0(\boldsymbol{x}) && \text{(unshifted coefficient)},\\ Z^{(2,2)}_{T^2}\big|_{q\to 0,\,\boldsymbol{x}=\mathbf{1}} &= y^{-d/2}\chi_{-y}(X) && \text{(compact Calabi–Yau target)},\\ Z^{(2,2)}_{T^2}\big|_{q\to 0,\,y=1,\,\boldsymbol{x}=\mathbf{1}} &= \chi(X)=\mathrm{Tr}(-1)^F && \text{(Witten index)}. \end{aligned} \tag{I.4.103}$$

Here the uncentered Hirzebruch convention is $\chi_t(X)=\sum_{p,q=0}^{d}(-1)^q h^{p,q}(X)t^p$; the substitution $t=-y$ supplies the extra $(-1)^p$, and $y^{-d/2}$ centers the R charges. Thus this is not an unshifted χ_y equality. For K3, the nonzero Hodge numbers are $h^{0,0}=h^{2,0}=h^{0,2}=h^{2,2}=1$ and $h^{1,1}=20$, so $\chi_{-y}(K3)=2+20y+2y^2$ and its centered value is $2y^{-1}+20+2y$. At $y=1$ it gives 24; weighting K3 only by a trivial flavor instead also gives the constant 24, with no y dependence. The distinction matters when a $\mathcal{N}=(0,2)$ theory carries nontrivial bundle data. Matching the regulated genus as a function is a stronger test than matching one number: two theories can share a Witten index yet differ in genus, and equality of the genus compares the signed protected spectrum. This is why it is the sharpest triality test in §I.4.8.

The free-field building blocks. The genus of an interacting theory is assembled from free-field factors, one per multiplet, and the two $\mathcal{N}=(0,2)$ multiplets contribute reciprocal factors. A free $\mathcal{N}=(0,2)$ chiral of fugacity z contributes

$$Z_\Phi \;=\; \frac{i\,\eta(\tau)}{\theta_1(\tau,z)}, \tag{I.4.104}$$

θ_1 in the *denominator*, a pole reflecting the boson zero modes. A free $\mathcal{N}=(0,2)$ Fermi of fugacity z contributes

$$Z_\Lambda \;=\; \frac{i\,\theta_1(\tau,z)}{\eta(\tau)}, \tag{I.4.105}$$

θ_1 in the *numerator*, a zero reflecting the left-moving fermion zero mode. With *matched* fugacity and R-shift assignments the two are reciprocal up to the phase,

$$Z_\Phi\, Z_\Lambda \;=\; \frac{i\,\eta}{\theta_1}\cdot\frac{i\,\theta_1}{\eta} \;=\; i^2 \;=\; -1, \tag{I.4.106}$$

θ_1 and η cancelling. This is a useful matched-argument cancellation, but it is only a *mnemonic*, not the general genus of a $\mathcal{N}=(2,2)$ chiral. In the $\mathcal{N}=(2,2)\to\mathcal{N}=(0,2)$ decomposition the chiral and Fermi factors usually carry R/flavor-shifted arguments, so their product is the standard nontrivial theta-function ratio; only when the arguments

are deliberately matched do the pole and zero cancel as in (I.4.106). The universal statement is the *placement*: chiral is θ_1-in-the-denominator, Fermi is θ_1-in-the-numerator, never swapped; for noncompact free chirals the boson zero modes add another subtlety.

The q-expansions make the matched-argument reciprocity concrete. In a schematic oscillator limit with the zero-mode prefactors suppressed, the chiral factor is the inverse product (the right-moving boson tower) and the Fermi factor is the product itself (the alternating single left-moving fermion sum),

$$\begin{aligned} Z_\Phi \;&\sim\; \frac{1}{\prod_{n\geq 1}(1-q^n)} \;=\; 1+q+2q^2+3q^3+5q^4+\cdots\,, \\ Z_\Lambda \;&\sim\; \prod_{n\geq 1}(1-q^n) \;=\; 1-q-q^2+q^5+\cdots\,, \end{aligned} \tag{I.4.107}$$

the chiral coefficients the partition numbers $p(n) = 1, 1, 2, 3, 5, \ldots$, the Fermi coefficients the pentagonal-number signs of Euler's identity. Their product is 1 order by order,

$$Z_\Phi\, Z_\Lambda \;\sim\; \big(1+q+2q^2+\cdots\big)\big(1-q-q^2+\cdots\big) \;=\; 1+0\cdot q+0\cdot q^2+\cdots \;=\; 1, \tag{I.4.108}$$

the reciprocity read off the series. This computation is deliberately limited: it says that a chiral pole and a Fermi zero cancel when their arguments match. It does *not* say that the matter contribution of every $\mathcal{N} = (2, 2)$ theory is trivial. A compact $\mathcal{N} = (2, 2)$ Calabi–Yau sigma model has a full q-dependent elliptic genus, a weak Jacobi form under the stated hypotheses; its $q \to 0$ limit with the left R fugacity is $y^{-d/2}\chi_{-y}(X)$ in the convention above, and setting $y = 1$ there gives the coarser signed index.

The gauge-theory genus and its localization. For a gauge theory the elliptic genus is a contour integral over the gauge-holonomy fugacities u, the one-loop determinants of the matter and gauge multiplets assembled into a meromorphic form and integrated by the Jeffrey–Kirwan prescription,

$$Z_{T^2} \;=\; \frac{1}{|W|}\oint_{\mathrm{JK}} \prod_a \frac{du_a}{2\pi i\, u_a}\; Z_{\mathrm{vec}}(u,\tau) \prod_{\mathrm{matter}} Z_{\Phi/\Lambda}(u,z,\tau), \tag{I.4.109}$$

the chiral (I.4.104) and Fermi (I.4.105) factors of the charged matter dressed with the gauge fugacities, Z_{vec} the Cartan factor, $|W|$ the Weyl-group order. The Jeffrey–Kirwan residue prescription selects which poles contribute (a chamber-dependent rule for the meromorphic holonomy integral). The output inherits modular covariance under the anomaly and regulator assumptions; holomorphy needs the additional spectral conditions above. The localization derivation and contour prescription belong to Volume IV.

Gauge consistency and holomorphy are separate conditions. The theta-function transformations detect the gauge anomaly. Without its cancellation the integrand does not define the gauge-invariant torus partition function of this stand-alone model. This necessary condition has a more limited consequence than holomorphy:

$$\begin{aligned} &\mathcal{A}_{\mathrm{gauge}} = 0 \;\not\Longrightarrow\; Z_{T^2} \text{ a holomorphic weak Jacobi form}, \\ &\mathcal{A}_{\mathrm{gauge}} \neq 0 \;\Longrightarrow\; \text{no gauge-invariant genus of the stated model.} \end{aligned} \tag{I.4.110}$$

For the noncompact $\mathbb{C}^4$ brick, $4-3-1=0$ tests the local nonabelian anomaly; it does not compactify the target or remove continuum and zero-mode effects. The free-chiral factor (I.4.104) already exhibits a pole at zero flavor fugacity. The compactness, charge and spectral hypotheses, or a specified equivariant regulator, are therefore part of any modularity or triality-index comparison.

I.4.10 $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,2)$: structures and uses

With both worlds built, a side-by-side comparison makes the architecture explicit and shows where each structure enters later constructions. Table I.4.3 reads the two worlds together: the balanced four-supercharge $\mathcal{N}=(2,2)$ theory with its two holomorphic sectors and Kähler geometry, against the chiral two-supercharge $\mathcal{N}=(0,2)$ theory with its independent E/J data and holomorphic bundle.

feature	$\mathcal{N}=(2,2)$	$\mathcal{N}=(0,2)$
supercharges	4 (two right, two left)	2 (two right)
holomorphic data	$W(\Phi)$ and $\widetilde{W}(\Sigma)$, two sectors	E_a and J^a, two independent data per Fermi
geometric meaning	Kähler / complex geometry, GLSM phases	holomorphic bundle $E\to X$, heterotic data
anomaly role	axial anomaly gates the B-twist / CY condition	gauge anomaly must cancel, gravitational anomaly only matches
extremization	inherited as the special case $E=TX$	c-extremization central, $c_R=3\,\mathrm{Tr}[\gamma^3R^2]$
protected index	left-R-graded genus; $q^0=y^{-d/2}\chi_{-y}(X)$ for compact CY X	flavor-graded genus; q^0 depends on bundle and flavor data
duality	mirror symmetry ($A(X)\cong B(\hat{X})$)	triality (order three)
later use	GLSM / mirror / Calabi–Yau phases (Volume IV)	brane brick models / $\mathcal{N}=(0,2)$ quivers (Volume VI)

Table I.4.3: The two $2d$ four-supercharge and two-supercharge worlds side by side. The $\mathcal{N}=(2,2)$ theory is recovered from $\mathcal{N}=(0,2)$ by locking $E=TX$; the genuinely $\mathcal{N}=(0,2)$ physics is the freedom $E\neq TX$, the independent E/J data, and the order-three triality. The elliptic genus is the shared protected object; under the stated spectral hypotheses it has Jacobi-form behavior. In a compact $\mathcal{N}=(2,2)$ Calabi–Yau sigma model, the left-R-graded q^0 term is the centered geometric genus. The two rightmost routings name the construction volumes that read this data: Volume IV for the $\mathcal{N}=(2,2)$ mirror and phase machinery, Volume VI for the $\mathcal{N}=(0,2)$ brane brick models.

Exit checklist

After this chapter the reader can

1. place a $2d$ theory on the supercharge count ($\mathcal{N}=(2,2)=4$, $\mathcal{N}=(0,2)=2$ real supercharges by $p+q$), identify the $\mathcal{N}=(2,2)$ chiral, twisted-chiral, and vector multiplets and the $\mathcal{N}=(0,2)$ chiral, Fermi, and vector multiplets, decompose $\mathcal{N}=(2,2)$ chiral $=\mathcal{N}=(0,2)$ chiral$+\mathcal{N}=(0,2)$ Fermi component by component, and name the vector and axial R-symmetries $U(1)_V$, $U(1)_A$;

2. (Part I) hold the two $\mathcal{N}=(2,2)$ holomorphic sectors apart (the ordinary $W(\Phi)$ over $\int d^2\theta$ and the twisted $\widetilde{W}(\Sigma)$ over $\int d\theta^+ d\bar{\theta}^-$), read the FI/theta $t=r-i\theta$ as the coefficient of the linear $\widetilde{W}=t\Sigma$, and name the chiral (Jacobi) and twisted-chiral (quantum-cohomology) rings;

3. (Part I) compute the $\mathbb{CP}^{N-1}$ quantum cohomology from the effective twisted superpotential $\widetilde{W}_{\text{eff}}=-t\Sigma-N\Sigma(\log\Sigma-1)$, get $\Sigma^N=e^{-t}=q$ and the ring $\mathbb{C}[\Sigma]/(\Sigma^N-q)$ with N vacua, and work the quintic in both chambers (the $\mathbb{P}^4$ quotient of dimension 4, the $G_5=0$ cut to 3, the $U(1)\to\mathbb{Z}_5$ Higgsing at $|P|^2=-r/5$, the matched $\hat{c}=3$) without confusing $W=0$ with the quotient-then-cut construction;

4. (Part I) name the A-twist ($U(1)_V$, quantum cohomology $\mathbb{C}[x]/(x^N-q)$, not gated by the CY condition) and the B-twist ($U(1)_A$, the Jacobi ring $\mathbb{C}[\Phi_i]/(\partial_i W)$, gated by $c_1(TX)=0$), and read the Hori–Vafa mirror exchanging them at the level of the untwisted theories;

5. (Part II) write the $\mathcal{N}=(0,2)$ scalar potential $V\supset\sum_a|E_a|^2+\sum_a|J^a|^2+V_D$, read its supersymmetric vacua $E_a=0$, $J^a=0$, verify the closure $\sum_a\mathrm{Tr}(E_aJ^a)=0$ (the Jacobi identity on the $\mathbb{C}^4$ brick), and state correctly that E and J enter symmetrically, distinguish complex conjugation of values from Fermi replacement with exchanged holomorphic E/J roles;

6. (Part II) distinguish an ungauged $\mathcal{N}=(2,2)$ LG model ($E=0$, $J=\partial W$) from a gauged $\mathcal{N}=(2,2)$ GLSM ($E_i=\sqrt{2}\,Q_i\,\Sigma\,\Phi_i\neq 0$), and verify the gauged quintic closure $\sum_i E_iJ^i+E_PJ^P=\sqrt{2}\,\Sigma P(\sum_i\phi_i\partial_iG_5-5G_5)=0$ by Euler homogeneity and gauge invariance (in this hypersurface the same degree-five charge assignment is also the Calabi–Yau condition), plus the bundle interpretation ($E=TX$ for $\mathcal{N}=(2,2)$, $E\neq TX$ for $\mathcal{N}=(0,2)$, with $\mathrm{ch}_2(E)=\mathrm{ch}_2(TX)$, $c_1(E)\equiv c_1(TX)\bmod 2$);

7. (Part II) check the $2d$ gauge anomaly ($4-3-1=0$) and the gravitational anomaly $c_R-c_L=\mathrm{Tr}\,\gamma^3$ (matched, not cancelled), and run c-extremization: assemble $c_R^{\text{tr}}=3\mathrm{Tr}[\gamma^3R^2]$, solve $k_{RF}=0$, get the positive $c_R=3k/(k+2)$ of the $\mathcal{N}=(0,2)$ A_{k+1} minimal model, read $c_L=c_R-\mathrm{Tr}\,\gamma^3$, and recognize the saddle (indefinite Hessian) and the negative-output formal example for what they are;

8. (Part II) follow the sourced $\mathcal{N} = (0,2)$ triality proposal in the declared flavor conventions (the $U(N_c)$ node with (N_1, N_2, N_3), the rank map $N_c \mapsto N_2 - N_c$, the flavor permutation, the dual matter with mesons and singlet and determinant Fermis), verify the order-three return $T^3 = \mathbb{1}$, and evaluate the local gauge and gravitational coefficients in the three frames, with the elliptic genus the sharpest check and the symmetric flavor quantity a toy consistency test;

9. (Part III) define the elliptic genus as a protected index (right-movers cancel), assign the chiral $(i\eta/\theta_1)$ and Fermi $(i\theta_1/\eta)$ factors (never swapped, reciprocal only for matched fugacities), write the Jeffrey–Kirwan localization skeleton, state its gauge-anomaly, spectral and regulator hypotheses, and read the $\mathcal{N} = (2,2)$-versus-$\mathcal{N} = (0,2)$ comparison table with its links to Volumes IV and VI.

Status and sources

Scope of the calculations. The polynomial quotient, Euler identities, trace Jacobi identity, conditional derivative, constrained polynomial calculation, structural charge equation and rank-orbit calculation are mathematical deductions [MATH] under the hypotheses stated with their equations. The computed anomaly coefficients [ANOM] are conditional on the stated representations. Matching one gravitational coefficient across the three frames does not establish duality.

Primary sources. Gadde–Gukov–Putrov (2013), arXiv:1310.0818, §2 after (3) supplies the Fermi replacement; §3.1's two matter tables and (48) fix the SQCD model and cyclic roles; §4, (63)–(65) and §4.1 fix local nonabelian and determinant anomalies and the removal of massive chiral–Fermi pairs. Benini–Bobev (2013), arXiv:1211.4030, (2)–(7) and (13), gives the anomaly-trace and stationarity premises under the normalizability hypotheses stated there. Benini–Eager–Hori–Tachikawa (2013), arXiv:1308.4896, §2.2, (2.13)–(2.15), supplies the determinants, and §2.3 supplies the residue prescription. Benini, Eager, et al. (2015), §2.1 (2.1) and §2.2 (2.12), distinguishes the left R fugacity of a $\mathcal{N} = (2,2)$ trace from general $\mathcal{N} = (0,2)$ flavor gradings; §4.1 (4.4) and §4.5 support the compact geometric limit used in. The K3 Hodge-number calculation following (I.4.103) fixes its sign and degree shift, but does not prove a full elliptic-genus identity. Murthy (2014), §1 and §3, gives the continuous-spectrum obstruction. The one-loop potential is imported from Hori–Vafa (2000), arXiv:hep-th/0002222, (3.39), with $Q_i = 1$ and $\mu = 1$. Differentiating it does not compute the loop determinant or prove its sign. The general supersymmetry condition comes from Gadde–Gukov–Putrov, §2, (4). Evaluating the brick examples does not derive the general condition.

The chapter uses the stated superpotential and $\mathcal{N} = (0,2)$ E/J conventions, the definition of GLSM charges as the integer kernel of the toric ray matrix, and the FI/theta parameter $t = r - i\theta$ and its chambers. It takes the gauge group to be $\prod_i U(N_i)$ in the indicated quivers. The (p,q) count and $2d$ Majorana–Weyl reality use the conventions of Chapter 1.

Part I introduces $\mathcal{N} = (2,2)$ in §I.4.2–§I.4.4, Part II treats $\mathcal{N} = (0,2)$ in §I.4.5–§I.4.8, and Part III discusses their shared tools in §I.4.9–§I.4.10.

§I.4.1 Kinematics and R-symmetries. The (p,q) supercharge count (I.4.1) with $\mathcal{N} = (2,2) = 4$ (I.4.2) and $\mathcal{N} = (0,2) = 2$ (I.4.4) (the reduction of $4d\,\mathcal{N} = 1$); the vector and axial R-symmetries $U(1)_V$, $U(1)_A$ (I.4.5) (the axial anomaly $\propto c_1$); the $\mathcal{N} = (2,2)$ chiral, twisted-chiral, and

vector multiplets, the $\mathcal{N}=(0,2)$ chiral, Fermi, and vector, and the $\mathcal{N}=(2,2)\to\mathcal{N}=(0,2)$ decompositions (I.4.9), (I.4.10) (Table I.4.1); the chapter map into the three parts. Source account (Witten (1993) the $\mathcal{N}=(2,2)$ multiplet grammar; Hori, Katz, et al. (2003) the $2d$ multiplet spine). Additional calculations include the (p,q) count $\mathcal{N}=(2,2)=4$, $\mathcal{N}=(0,2)=2$ from the ch.1 ladder, the $\mathcal{N}=(2,2)\to\mathcal{N}=(0,2)$ content bookkeeping, and that the Fermi supplies the left-moving fermion the $\mathcal{N}=(0,2)$ chiral lacks.

§I.4.2 Part I: the $\mathcal{N}=(2,2)$ holomorphic sectors and rings. The two holomorphic data of a $\mathcal{N}=(2,2)$ theory: the ordinary $W(\Phi)$ over $\int d^2\theta$ (I.4.11) and the twisted $\widetilde{W}(\Sigma)$ over $\int d\theta^+ d\bar{\theta}^-$ (I.4.12), with the linear $\widetilde{W}=t\Sigma$, $t=r-i\theta$ (I.4.13) carrying the FI/theta parameter as twisted-chiral data (I.4.14); the chiral (Jacobi) ring $\mathcal{R}_B=\mathbb{C}[\Phi_i]/(\partial_i W)$ (I.4.15) and the twisted-chiral (quantum-cohomology) ring, exchanged by mirror symmetry, with the worked A_{k+1} Jacobi ring (I.4.16)–(I.4.18). Source account (Witten (1993) the $\mathcal{N}=(2,2)$ grammar and FI/theta; Hori, Katz, et al. (2003) the two rings). The identification of $\mathbb{C}[\Phi_i]/(\partial_i W)$ with the physical LG chiral ring is imported from the cited sources. The worked one-variable quotient uses elementary polynomial division. Using the supplied $\mathbb{CP}^{N-1}$ relation Σ^N-q determines the quotient and its count in that example. The physical ring identification and a general Jacobi-quotient construction require additional input.

§I.4.3 Part I: the GLSM, $\mathbb{CP}^{N-1}$, the quintic. The gauged linear sigma model, the D-term $\sum_i Q_i^a|\phi_i|^2=r_a$ (I.4.19), the $\mathbb{CP}^{N-1}$ effective twisted superpotential $\widetilde{W}_{\text{eff}}=-t\Sigma-N\Sigma(\log\Sigma-1)$ (I.4.23), the vacuum equation $\partial_\Sigma\widetilde{W}_{\text{eff}}=-t-N\log\Sigma=0\Rightarrow\Sigma^N=e^{-t}=q$ (I.4.24), and the quantum cohomology $\mathbb{C}[\Sigma]/(\Sigma^N-q)$ with N vacua (I.4.25); the FI chambers, the quantum Kähler moduli space with its discriminant locus (a massless state / noncompact Coulomb branch, not a shrinking curve), the geometric (GIT quotient / Calabi–Yau) and Landau–Ginzburg phases as two regimes of one RG flow, and the CY condition $\sum_i Q_i^a=0$ (I.4.29); the quintic walked ($U(1)$ with $(+1)^5,-5$, $W=PG_5$ (I.4.30), the $\mathbb{P}^4$ quotient of dimension 4 (I.4.33), the $G_5=0$ cut to 3 (I.4.34), the $U(1)\to\mathbb{Z}_5$ Higgsing at $|P|^2=-r/5$ (I.4.35), and $\hat{c}=3$ geometric and LG (I.4.36)–(I.4.37)). Source account (Witten (1993) the phases, the FI chambers, the quintic; Hori and Vafa (2000) the effective twisted superpotential and mirror; Witten (1994) the LG central charge). Additional calculations include the $\mathbb{CP}^{N-1}$ vacuum equation $\partial_\Sigma\widetilde{W}_{\text{eff}}=-t-N\log\Sigma$ (the sign giving $\Sigma^N=e^{-t}$, not the plus-sign e^{+t}), the N-vacua quantum cohomology, the quintic $\sum Q^a=0$, the $\mathbb{P}^4$ quotient dimension, the hypersurface dimension 3, the $U(1)\to\mathbb{Z}_5$ Higgsing, and the LG $\hat{c}=\sum(1-2q_i)=3$ matching the geometric $\hat{c}=3$. The phase theorem is routed.

§I.4.4 Part I: the A and B twists and mirror symmetry. The two $\mathcal{N}=(2,2)$ topological twists: the A-twist $(U(1)_V$, holomorphic maps, quantum cohomology $\mathcal{R}_A=\mathbb{C}[x]/(x^N-q)$ (I.4.40) tied to §I.4.3, NOT gated by the CY condition) and the B-twist $(U(1)_A$, constant maps, the Jacobi ring $\mathcal{R}_B=\mathbb{C}[\Phi_i]/(\partial_i W)$ (I.4.41), requires $c_1(TX)=0$ (I.4.42), i.e. $\sum_i Q_i=0$ for a GLSM (I.4.43)); mirror symmetry as a duality of the full untwisted $\mathcal{N}=(2,2)$ theories, realized by the Hori–Vafa GLSM mirror (I.4.45)–(I.4.47), exchanging $A(X)\cong B(\hat{X})$ (I.4.48) and so $\mathcal{R}_A(X)\cong\mathcal{R}_B(\hat{X})$, the twists how the duality is used, not a prerequisite. Source account (Witten (1993) the twists and the CY condition; Hori and Vafa (2000) the mirror realization). Additional calculations include the B-twist CY gating $\sum Q=0$ (allowed on the quintic, blocked on the non-CY $(+1)^5,-4$) and the A-twist NOT gated by the CY condition, and the $\mathbb{CP}^{N-1}$ quantum-cohomology ring tied to (I.4.40). The topological-correlator / mirror-theorem machinery is routed to Volume IV.

§I.4.5 Part II: the $\mathcal{N}=(0,2)$ E/J data and scalar potential. The two holomorphic data $\bar{D}_+\Lambda_a=E_a$ (I.4.49) and $W_{\mathcal{N}=(0,2)}=\int d\theta^+\sum_a\Lambda_a J^a$ (I.4.50), the scalar potential $V\supset$

$\sum_a |E_a|^2 + \sum_a |J^a|^2 + V_D$ (I.4.51) with the SUSY vacua $E_a = 0$, $J^a = 0$ (I.4.52); the precise statement that E and J enter symmetrically but conjugation complex-conjugates values, whereas conjugate-representation Fermi replacement exchanges holomorphic E/J roles with the plus convention (I.4.53); the closure $\sum_a \mathrm{Tr}(E_a J^a) = 0$ (I.4.54) as the Jacobi identity on the $\mathbb{C}^4$ brick commutators (I.4.56)–(I.4.57) (with the wrong pairing nonzero (I.4.58)), inherited by the $\mathbb{C}^4/\mathbb{Z}_4$ orbifold (4/16/12 (I.4.60)). Source account (Distler and Kachru (1994) the $\mathcal{N} = (0,2)$ E/J / Landau–Ginzburg structure; Franco et al. (2016) the brick E/J and $\sum \mathrm{Tr}(EJ) = 0$). Additional calculations include E and J are two independent data, $\sum_a \mathrm{Tr}(E_a J^a) = 0$ on non-commuting matrices (the $\mathbb{C}^4$ Jacobi identity) and its orbifold descent, and the 4/16/12 count.

§I.4.6 Part II: $\mathcal{N} = (0,2)$ bundles and the gauged E-term. The bundle interpretation ($E = TX$ for $\mathcal{N} = (2,2)$, an independent bundle for $\mathcal{N} = (0,2)$, with $\mathrm{ch}_2(E) = \mathrm{ch}_2(TX)$, $c_1(E) \equiv c_1(TX)$ mod 2 (I.4.61)–(I.4.62)); the ungauged $\mathcal{N} = (2,2)$ LG specialization $E = 0$, $J^a = \partial W/\partial\Phi_a$ (I.4.63)–(I.4.64); the correction that a GAUGED $\mathcal{N} = (2,2)$ GLSM has $E_i = \sqrt{2}\, Q_i\, \Sigma\, \Phi_i \neq 0$ (I.4.65), with the worked gauged quintic $E_i = \sqrt{2}\,\Sigma\phi_i$, $E_P = -5\sqrt{2}\,\Sigma P$, $J^i = P\partial_i G_5$, $J^P = G_5$ (I.4.66) and its closure $\sum_i E_i J^i + E_P J^P = \sqrt{2}\,\Sigma P(\sum_i \phi_i \partial_i G_5 - 5G_5) = 0$ by gauge invariance / quasi-homogeneity (I.4.67)–(I.4.68) (in this hypersurface the same degree-five charge assignment is also the Calabi–Yau condition), and the $\mathcal{N} = (0,2)$ deformation with $E \neq TX$. Source account (Distler and Kachru (1994) the $\mathcal{N} = (0,2)$ LG / GLSM E-term and bundle anomalies; Witten (1993) the gauged linear sigma model). Additional calculations include the gauged quintic E/J closure $\sum_i E_i J^i + E_P J^P = 0$ by Euler homogeneity, COMPUTED on a six-monomial degree-five fixture G_5 in SymPy, together with the Euler identity $\sum_i \phi_i \partial_i G_5 = 5G_5$, and the failure of closure for a polynomial G of the wrong degree. The monomial proof gives the general degree-five Euler identity.

§I.4.7 Part II: anomalies and c-extremization. The gauge anomaly $\mathrm{Tr}_{\mathrm{chiral}} - \mathrm{Tr}_{\mathrm{Fermi}} - \mathrm{Tr}_{\mathrm{adj}} = 0$ (I.4.70) (the brick $4-3-1=0$ (I.4.71), automatic for $\mathcal{N} = (2,2)$ (I.4.72), the GGP nonabelian coefficient for $N_c \geq 2$, $N_1 + N_2 - N_3 - 2N_c = 0$ (I.4.73)) and the gravitational anomaly $c_R - c_L = \mathrm{Tr}\,\gamma^3$ (I.4.75) (generically nonzero, the $\mathbb{C}^4/\mathbb{Z}_4$ $16-12-4=0$ (I.4.76), matched, not cancelled); c-extremization $c_R^{\mathrm{tr}} = 3\mathrm{Tr}[\gamma^3 R^2]$ (I.4.78), the stationarity $k_{RF} = 0$ (I.4.79), $c_L = c_R - \mathrm{Tr}\,\gamma^3$ (I.4.80), a STATIONARY point (a saddle, the indefinite Hessian (I.4.81)–(I.4.82)), NOT a maximum, with the hypotheses stated (candidate IR SCFT, anomaly-free trial R, no accidental symmetries); the POSITIVE flagship $\mathcal{N} = (0,2)$ A_{k+1} minimal model $c_R = 3k/(k+2)$ ($k = 2 \Rightarrow c_R = 3/2$) (I.4.83)–(I.4.86), the deliberately-formal saddle fixture $c_R = -0.36$ (I.4.87)–(I.4.89) (outside the unitary SCFT domain), and the symmetric brick $R_X = 1/2$ (I.4.90). Source account (Benini and Bobev (2013a) c-extremization; Benini and Bobev (2013b) the companion; Distler and Kachru (1994) the $\mathcal{N} = (0,2)$ gauge anomaly; Gadde et al. (2014) the GGP node anomaly). Additional calculations include the brick $4-3-1=0$ with the vector essential, the GGP nonabelian rank polynomial (not the full $U(N_c)$ anomaly), the gravitational anomaly matched across two descriptions, the $k_{RF} = 0$ solve, the POSITIVE minimal-model $c_R = 3k/(k+2) = 3\hat{c}$ (positive for every k), the formal fixture's negative c_R flagged NOT unitary, the indefinite (saddle) Hessian eigenvalues $\{+6, -6\}$, and the forced brick $R_X = 1/2$.

§I.4.8 Part II: $\mathcal{N} = (0,2)$ triality. The order-three duality (the $m = 2$ rung of the graded-quiver ladder, entry 18 signposted) in the declared representation conventions; the $U(N_c)$ node with N_1 fundamental chirals, N_2 antifundamental chirals, N_3 fundamental Fermis (I.4.91), balanced rank $N_c = \frac{1}{2}(N_1 + N_2 - N_3)$ (I.4.92), the rank map $N_c \mapsto N_2 - N_c$ (I.4.94) with the flavor permutation (I.4.95), the dual matter (charged Ψ distinguished from singlet Γ, the two determinant Fermis, surviving Fermi mesons $\Gamma' = \Phi\Psi$, and integrated-out (M, Γ) pairs), the

order-three return $(5,3,4) \to (3,4,5) \to (4,5,3) \to (5,3,4)$ (I.4.98), $T^3 = \mathbb{1}$ (Table I.4.2). The load-bearing evidence is the anomaly and elliptic-genus matches across the three frames; the symmetric flavor quantity I (I.4.100) is a toy consistency check (a permutation-symmetric polynomial in the ranks, useful only to reject a wrong-multiset candidate), NOT a named central-charge or anomaly formula. Source account (Gadde et al. (2014) the $2d$ $\mathcal{N} = (0,2)$ triality, the rank map, the dual matter). The rank calculation evaluates each frame's balanced-rank polynomial, the order-three return (not order two), the rank-map-only order-two involution, the symmetric point as a triality fixed point, and the triality-preserved flavor invariant. The abelian and gravitational coefficient calculations also use the full stated spectrum. The calculations are consistency conditions for the proposed infrared duality. The further dynamical discussion is routed. $\mathcal{N} = (0,2)$ triality has the cyclic matter and rank transformations shown above.

§I.4.9 Part III: the elliptic genus. The general flavor-graded $\mathcal{N} = (0,2)$ trace and the left-R-graded $\mathcal{N} = (2,2)$ trace (I.4.102) are protected under discrete-spectrum and convergence hypotheses; weak-Jacobi-form behavior additionally needs the charge-lattice, holomorphy and Fourier-growth hypotheses stated above. The former has unshifted coefficient $[q^0]Z_{T^2}^{(0,2)} = I_0(\boldsymbol{x})$, which may vanish; extracting a leading ground contribution requires its specified q-power. For a compact $\mathcal{N} = (2,2)$ Calabi–Yau target the latter gives $y^{-d/2}\chi_{-y}(X)$ in the stated Hirzebruch convention and $\chi(X)$ at $y = 1$ (I.4.103). The free chiral factor $i\eta/\theta_1$ (DENOMINATOR) (I.4.104) and the free Fermi factor $i\theta_1/\eta$ (NUMERATOR) (I.4.105), reciprocal only for matched arguments (I.4.106) (a mnemonic, not the general $\mathcal{N} = (2,2)$ chiral genus: R/flavor shifts usually make a nontrivial theta-function ratio, and noncompact free chirals have subtle zero modes), with q-expansions $1/\prod(1-q^n)$ and $\prod(1-q^n)$ (I.4.107); the Jeffrey–Kirwan localization skeleton (I.4.109) (named, routed); modularity gated by the gauge anomaly (I.4.110). Source account (Benini, Eager, et al. (2014) and Benini, Eager, et al. (2015) the elliptic genus by localization, the JK residue, the Jacobi-form structure). Additional calculations include the chiral $= \theta_1$-denominator / Fermi $= \theta_1$-numerator placement (never swapped), the reciprocal product $= -1$, the q-expansion leading coefficients and their reciprocal series, and the necessary anomaly condition ($4 - 3 - 1 = 0$ versus $5 - 3 - 1 \neq 0$). That calculation tests a necessary condition, not modularity, holomorphy or a determinant. The full index machinery (localization, the JK contour, modularity, duality invariance) is routed.

§I.4.10 Part III: the $\mathcal{N} = (2,2)$-versus-$\mathcal{N} = (0,2)$ comparison. The side-by-side comparison table (Table I.4.3) reading the two worlds together (supercharges, holomorphic data, geometric meaning, anomaly role, extremization, protected index, duality, engine use) and routing each structure to its construction volume: the $\mathcal{N} = (2,2)$ mirror and phase machinery to Volume IV, the $\mathcal{N} = (0,2)$ brane brick models to Volume VI. A summary section, no new claim; it recapitulates the structures of Parts I and II.

Routed (stated, not proved). The c-extremization theorem, the $2d$ c-theorem, the accidental-symmetry / unitarity-bound corrections, and the dynamics of the $\mathcal{N} = (0,2)$ triality proposal beyond its necessary anomaly / index tests; the gauged-linear-sigma-model phase theorem (the full quantum-Kähler-moduli / secondary-fan / discriminant analysis), the elliptic-genus localization / Jeffrey–Kirwan / modularity / duality machinery, and the A/B-twist topological-correlator / mirror-theorem machinery (Volume IV). All are stated and, where a computation is possible at toolkit depth, run on fixtures; their proofs live in Part III and Volume IV.

Further reading

The gauged linear sigma model and its phases are developed in Witten (1993); the original geometric mirror statement is Greene and Plesser (1990), and quantum cohomology of toric targets is in Morrison and Plesser (1995). The tt^* geometry and the classification of two-dimensional $\mathcal{N} = 2$ theories appear in Cecotti and Vafa (1991) and Vafa (1991), and the Bethe/gauge correspondence behind the vacuum equations in Nekrasov and Shatashvili (2009b) and Nekrasov and Shatashvili (2009a). For $(0, 2)$ models see Silverstein and Witten (1994); the two-dimensional conformal minimal models are in Friedan et al. (1984), with the textbook Di Francesco et al. (1997).

For modern extensions, global forms, discrete theta angles, and nonabelian mirrors of pure $\mathcal{N} = (2, 2)$ gauge theories are analyzed in Gu et al. (2021), new $\mathcal{N} = (0, 2)$ dualities and their elliptic-genus tests in Sacchi (2020), and the related decomposition phenomenon of two-dimensional gauge theories is reviewed in Sharpe (2024).

References

Benini, F. and N. Bobev (2013a). "Exact two-dimensional superconformal R-symmetry and c-extremization". In: *Phys. Rev. Lett.* 110, p. 061601. arXiv: 1211.4030 [hep-th].

Benini, F. and N. Bobev (2013b). "Two-dimensional SCFTs from wrapped branes and c-extremization". In: *JHEP* 06, p. 005. DOI: 10.1007/JHEP06(2013)005. arXiv: 1302.4451 [hep-th].

Benini, F., R. Eager, et al. (2014). "Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups". In: *Lett. Math. Phys.* 104, pp. 465–493. DOI: 10.1007/s11005-013-0673-y. arXiv: 1305.0533 [hep-th].

Benini, F., R. Eager, et al. (2015). "Elliptic Genera of 2d N=2 Gauge Theories". In: *Commun. Math. Phys.* 333.3, pp. 1241–1286. DOI: 10.1007/s00220-014-2210-y. arXiv: 1308.4896 [hep-th].

Cecotti, S. and C. Vafa (1991). "Topological antitopological fusion". In: *Nucl. Phys. B* 367, pp. 359–461. DOI: 10.1016/0550-3213(91)90021-O.

Di Francesco, P., P. Mathieu, and D. Senechal (1997). *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag. DOI: 10.1007/978-1-4612-2256-9.

Distler, J. and S. Kachru (1994). "(0,2) Landau-Ginzburg theory". In: *Nucl. Phys. B* 413, pp. 213–243. DOI: 10.1016/0550-3213(94)90619-X. arXiv: hep-th/9309110.

Franco, S., S. Lee, and R.-K. Seong (2016). "Brane Brick Models, Toric Calabi-Yau 4-Folds and 2d (0,2) Quivers". In: *JHEP* 02, p. 047. DOI: 10.1007/JHEP02(2016)047. arXiv: 1510.01744 [hep-th].

Friedan, D., Z.-a. Qiu, and S. H. Shenker (1984). "Conformal Invariance, Unitarity and Two-Dimensional Critical Exponents". In: *Phys. Rev. Lett.* 52, pp. 1575–1578. DOI: 10.1103/PhysRevLett.52.1575.

Gadde, A., S. Gukov, and P. Putrov (2014). "(0, 2) trialities". In: *JHEP* 03, p. 076. DOI: 10.1007/JHEP03(2014)076. arXiv: 1310.0818 [hep-th].

Greene, B. R. and M. R. Plesser (1990). "Duality in Calabi-Yau Moduli Space". In: *Nucl. Phys. B* 338, pp. 15–37. DOI: 10.1016/0550-3213(90)90622-K.

Gu, W., E. Sharpe, and H. Zou (2021). "Notes on two-dimensional pure supersymmetric gauge theories". In: *JHEP* 04, p. 261. DOI: 10.1007/JHEP04(2021)261. arXiv: 2005.10845 [hep-th].

Hori, K., S. Katz, et al. (2003). *Mirror symmetry*. Vol. 1. Clay Mathematics Monographs. Providence, USA: AMS.

Hori, K. and C. Vafa (2000). "Mirror symmetry". In: arXiv: hep-th/0002222.

Morrison, D. R. and M. R. Plesser (1995). "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties". In: *Nucl. Phys. B* 440, pp. 279–354. DOI: 10.1016/0550-3213(95)00061-V. arXiv: hep-th/9412236.

Murthy, S. (2014). "A holomorphic anomaly in the elliptic genus". In: *JHEP* 06, p. 165. DOI: 10.1007/JHEP06(2014)165. arXiv: 1311.0918 [hep-th].

Nekrasov, N. A. and S. L. Shatashvili (2009a). "Quantum integrability and supersymmetric vacua". In: *Prog. Theor. Phys. Suppl.* 177. Ed. by T. Takayanagi and S. Yahikozawa, pp. 105–119. DOI: 10.1143/PTPS.177.105. arXiv: 0901.4748 [hep-th].

Nekrasov, N. A. and S. L. Shatashvili (2009b). "Supersymmetric vacua and Bethe ansatz". In: *Nucl. Phys. B Proc. Suppl.* 192-193. Ed. by L. Baulieu et al., pp. 91–112. DOI: 10.1016/j.nuclphysbps.2009.07.047. arXiv: 0901.4744 [hep-th].

Sacchi, M. (2020). "New 2d $\mathcal{N} = (0, 2)$ dualities from four dimensions". In: *JHEP* 12, p. 009. DOI: 10.1007/JHEP12(2020)009. arXiv: 2004.13672 [hep-th].

Sharpe, E. (2024). "An introduction to decomposition". In: *2021–2022 MATRIX Annals*. Ed. by D. R. Wood, J. de Gier, and C. E. Praeger. Springer International Publishing. DOI: 10.1007/978-3-031-47417-0_8. arXiv: 2204.09117 [hep-th].

Silverstein, E. and E. Witten (1994). "Global U(1) R symmetry and conformal invariance of (0,2) models". In: *Phys. Lett. B* 328, pp. 307–311. DOI: 10.1016/0370-2693(94)91484-2. arXiv: hep-th/9403054.

Vafa, C. (1991). "Topological Landau-Ginzburg models". In: *Mod. Phys. Lett. A* 6, pp. 337–346. DOI: 10.1142/S0217732391000324.

Witten, E. (1993). "Phases of N=2 theories in two dimensions". In: *Nucl. Phys. B* 403, pp. 159–222. arXiv: hep-th/9301042.

Witten, E. (1994). "On the Landau-Ginzburg description of N=2 minimal models". In: *Int. J. Mod. Phys. A* 9, pp. 4783–4800. DOI: 10.1142/S0217751X9400193X. arXiv: hep-th/9304026.

Chapter I.5

$4d$ $\mathcal{N} = 2$ supersymmetric field theories

Reader contract. A gauge theory can become strongly coupled while some of its particles become light. Four-dimensional $\mathcal{N} = 2$ supersymmetry makes this phenomenon calculable: the same holomorphic periods determine the low-energy couplings and the masses of BPS states. A vanishing period identifies a light charged particle, and changing the electric-magnetic frame can turn a strongly coupled description into a weakly coupled one.

We develop this connection using the $\mathcal{N} = 1$ multiplets of Chapter 3. Eight real supercharges constrain their interactions, organize the Coulomb branch by a prepotential, and give the Higgs branch a hyperkähler geometry. Pure $SU(2)$ provides our main example. Its Seiberg–Witten curve locates the monopole and dyon singularities; its monodromies connect those singularities to the coupling at large scalar expectation value. The curve and the identification of its periods with physical charges are supplied by the Seiberg–Witten solution. We work out their consequences here; the microscopic derivation is developed in Part III.

What this chapter delivers

The calculations follow the path from fields to observables. We first pair the $\mathcal{N} = 1$ fields into $\mathcal{N} = 2$ multiplets and determine their canonical coupling (§I.5.1). Electromagnetic duality then explains how to compare charge and period coordinates in different frames (§I.5.2). Differentiating the prepotential gives the coupling and metric (§I.5.3), while the pure $SU(2)$ curve gives singularities, BPS masses, monodromies and the local monopole condensate (§I.5.4). Adding matter leads to scale matching and walls of marginal stability (§I.5.5).

The remaining examples explain what changes outside the pure Coulomb-branch problem: a hyperkähler quotient produces the Higgs branch (§I.5.6); the massless $N_f = 4$ theory has a marginal coupling and flavor triality (§I.5.7); free-field anomaly coefficients determine Lagrangian conformal central charges (§I.5.8); and a singular local curve yields the fractional Coulomb dimension of an Argyres–Douglas fixed point (§I.5.9).

I.5.1 $\mathcal{N}=2$ multiplets in $\mathcal{N}=1$ language

A four-dimensional $\mathcal{N}=2$ theory carries eight real supercharges. Chapter 1 records this count as the $8Q$ row of the dimensional table, the same amount as $5d$ $\mathcal{N}=1$ and $3d$ $\mathcal{N}=4$. Eight supercharges are twice the four of the $4d$ $\mathcal{N}=1$ theories of Chapter 3, so the vector and full hypermultiplets considered here each split into a pair of $\mathcal{N}=1$ multiplets under the $\mathcal{N}=1$ subalgebra. This is the working description used throughout the chapter. The second supersymmetry relates the two $\mathcal{N}=1$ pieces and constrains their interactions.

There are two basic multiplets. The $\mathcal{N}=2$ *vector multiplet* decomposes, under the $\mathcal{N}=1$ subalgebra, into an $\mathcal{N}=1$ vector multiplet V and an $\mathcal{N}=1$ adjoint chiral multiplet Φ,

$$\mathcal{N}=2 \text{ vector } = \underbrace{V}_{\mathcal{N}=1 \text{ vector}} \oplus \underbrace{\Phi}_{\mathcal{N}=1 \text{ adjoint chiral}} . \tag{I.5.1}$$

It contains a gauge field A_μ, a complex adjoint scalar ϕ (the scalar of Φ), and two Weyl gauginos, the gaugino of V together with the fermion of Φ. The second supersymmetry relates the two gauginos. They form a doublet of the $SU(2)_R$ that rotates them into each other. The $\mathcal{N}=2$ *hypermultiplet* decomposes into two $\mathcal{N}=1$ chiral multiplets in conjugate representations,

$$\mathcal{N}=2 \text{ hyper } = \; Q \;\oplus\; \widetilde{Q}, \qquad Q \in \mathbf{R}, \quad \widetilde{Q} \in \overline{\mathbf{R}}. \tag{I.5.2}$$

It carries four real scalars, the components of Q and $\widetilde{Q}$, which assemble into a quaternion, and two Weyl fermions. Per gauge-representation component, the complex pair $(Q, \widetilde{Q}^\dagger)$ transforms as an $SU(2)_R$ doublet. The names V, Φ, Q, $\widetilde{Q}$ are exactly the $\mathcal{N}=1$ multiplets of Chapter 3. What is new is the pairing rule, and the R-symmetry that enforces it.

The R-symmetry of $4d$ $\mathcal{N}=2$ is the master-table row of Chapter 1,

$$R \;=\; SU(2)_R \times U(1)_R, \qquad \dim R \;=\; 3+1 \;=\; 4, \qquad \operatorname{rank} R \;=\; 1+1 \;=\; 2. \tag{I.5.3}$$

The $SU(2)_R$ rotates the two supercharges of each chirality into one another, and the $U(1)_R$ acts on the adjoint scalar ϕ and grades the chiral ring. The two supercharges of a fixed chirality carry the defining doublet, so their continuous automorphisms have Lie algebra $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$. Equal rank does not identify this algebra with $\mathfrak{su}(3)$. In an asymptotically free quantum theory the continuous $U(1)_R$ can be anomalous, as the pure $SU(2)$ example below will show.

An $\mathcal{N}=2$ gauge theory is then built from these pieces: the vector multiplets gauge a group G, the hypermultiplets sit in a representation of G, and the second supersymmetry fixes the superpotential and gauge kinetic term in terms of $\mathcal{N}=1$ data. The new physics is not the field content but the eight-supercharge structure the $\mathcal{N}=1$ language misses: the rigid special geometry of the Coulomb branch, the curve that solves it, and the BPS central charge.

The canonical $\mathcal{N}=2$ coupling, in $\mathcal{N}=1$ words. The pairing rule also pins down the interactions. In $\mathcal{N}=1$ superspace, the adjoint chiral Φ couples to a hypermultiplet

$(Q, \widetilde{Q})$ through a superpotential the second supersymmetry fixes completely,

$$W_{\mathcal{N}=2} \;=\; \sqrt{2}\,\widetilde{Q}\,\Phi\,Q \;+\; m\,\widetilde{Q}\,Q. \tag{I.5.4}$$

Three facts make this $\mathcal{N}=2$ rather than a generic $\mathcal{N}=1$ superpotential. First, Φ sits in the adjoint of G and contracts the gauge indices of $Q \in \mathbf{R}$ and $\widetilde{Q} \in \overline{\mathbf{R}}$ into a singlet, the cubic coupling selected by the gauge action on $\mathbf{R}$,

$$\mathbf{R} \otimes \mathrm{adj} \otimes \overline{\mathbf{R}} \;\supset\; \mathbf{1}. \tag{I.5.5}$$

Second, the cubic coefficient is not free: the second supersymmetry rotates the gaugino of V into the fermion of Φ, tying the gauge coupling g to the Yukawa coupling in (I.5.4). The factor $\sqrt{2}$ is the normalization that makes the central charge come out as the fixed $Z = a\,n_e + a_D\,n_m$ of §I.5.4, not an adjustable parameter. Third, the mass term is the unique $\mathcal{N}=2$-preserving mass, giving equal masses to the two $\mathcal{N}=1$ chirals as the $SU(2)_R$ demands,

$$m_Q \;=\; m_{\widetilde{Q}} \;=\; m. \tag{I.5.6}$$

A generic change of these interactions breaks the second supersymmetry. The Yukawa coupling must remain tied to the gauge interaction; it is this relation that allows the low-energy vector-multiplet action to be described by one prepotential.

I.5.2 Electromagnetic duality and the charge lattice

On a generic vacuum the adjoint scalar ϕ acquires a VEV, the gauge group breaks to its Cartan $U(1)^r$, and the low-energy theory is abelian. An abelian gauge theory in four dimensions carries both electric and magnetic charges. Electric-magnetic changes of frame exchange which charges are treated as electric; they need not be symmetries of a fixed microscopic theory.

The low-energy coupling is packaged into one complex number, the complexified coupling

$$\tau \;=\; \frac{\theta}{2\pi} \;+\; \frac{4\pi i}{g^2}, \qquad \mathrm{Im}\,\tau \;=\; \frac{4\pi}{g^2} \;>\; 0, \tag{I.5.7}$$

whose real part is the theta-angle and whose imaginary part is the inverse gauge coupling squared. The positivity of $\mathrm{Im}\,\tau$ is the positivity of the gauge kinetic term. The electromagnetic-duality group is $SL(2,\mathbb{Z})$, generated by two transformations,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} :\; \tau \mapsto -\frac{1}{\tau}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} :\; \tau \mapsto \tau + 1. \tag{I.5.8}$$

The generator S exchanges electric and magnetic charges and inverts the coupling, T is the periodicity of the theta-angle $\theta \mapsto \theta + 2\pi$. Each acts on τ by the Möbius transformation of its period matrix. For a fixed physical state, charges transform by the inverse transpose, so their contraction with the period column is unchanged,

$$\tau \;\mapsto\; \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} n_m \\ n_e \end{pmatrix} \;\mapsto\; \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} n_m \\ n_e \end{pmatrix}. \tag{I.5.9}$$

Both generators have determinant one, and they obey the defining relations of $SL(2,\mathbb{Z})$,

$$\det S \;=\; \det T \;=\; 1, \qquad S^2 \;=\; -\mathbb{1}, \qquad (ST)^3 \;=\; -\mathbb{1}, \tag{I.5.10}$$

the small algebraic check that the duality is well defined.

The magnetically charged states are solitons: on the Coulomb branch the broken gauge symmetry supports a 't Hooft–Polyakov monopole, a finite-energy solution of one unit of magnetic charge, while a purely electric state is a quantum of the unbroken $U(1)$. Changing to a magnetic frame treats a soliton as an electrically charged excitation of the dual photon. This explains how the weak local descriptions at strong-coupling singularities can use fields absent from the original weak-coupling Lagrangian.

The rank-r charge lattice and $Sp(2r,\mathbb{Z})$. Rank one is the simplest case, not the general one. A theory of rank r breaks to $U(1)^r$, and a BPS state carries r electric and r magnetic charges. The charge vector lives in the electromagnetic lattice $\Gamma_{\rm em} \cong \mathbb{Z}^{2r}$,

$$\gamma \;=\; (n_{e,I},\, n_m^I) \;\in\; \Gamma_{\rm em}, \qquad I = 1,\dots,r. \tag{I.5.11}$$

The lattice carries a canonical antisymmetric integer pairing, the *Dirac pairing*,

$$\langle \gamma, \gamma' \rangle \;=\; n_{e,I}\, n_m'^I \;-\; n_m^I\, n'_{e,I} \;=\; \gamma^T J\, \gamma', \qquad J = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}. \tag{I.5.12}$$

The pairing is the integer counting the relative electric-magnetic charge of two states,

$$\langle \gamma, \gamma' \rangle \;=\; -\langle \gamma', \gamma \rangle, \qquad \langle \gamma, \gamma \rangle \;=\; 0, \qquad \langle e_I, m^J \rangle \;=\; \delta_I^{\,J}, \tag{I.5.13}$$

antisymmetric, self-pairing zero, basic electric against basic magnetic pairing to ± 1. We fix the ordering once: the prose labels every charge electric-first, $\gamma = (n_{e,I}, n_m^I)$, following (I.5.14). A duality matrix acting on the period vector transforms a fixed charge by the dual (contragredient) action, so the central charge Z_γ stays invariant; §I.5.4 acts the matrices on the period column $(a_D, a)^T$ and states its paired charge ordering (n_m, n_e) at the point of use. The low-energy data are packaged in the *period vector* $\Pi = (a_D^I, a_I)$, built from the special coordinates and their duals, and the central charge of a state of charge γ is the charge-period contraction

$$Z_\gamma \;=\; n_{e,I}\, a^I \;+\; n_m^I\, a_{D,I} \;+\; (\text{flavor-mass terms}), \tag{I.5.14}$$

the rank-r generalization of the rank-one $Z = a\, n_e + a_D\, n_m$ of §I.5.4 (the flavor masses are added in §I.5.5). The duality group is the group of lattice automorphisms that preserve the Dirac pairing, the integer symplectic group

$$M \in Sp(2r,\mathbb{Z}): \qquad M^T J\, M \;=\; J, \qquad M \in \mathrm{GL}(2r,\mathbb{Z}). \tag{I.5.15}$$

At rank one a 2×2 integer matrix preserves J exactly when its determinant is one, so the symplectic group collapses to the modular group,

$$Sp(2,\mathbb{Z}) \;=\; SL(2,\mathbb{Z}), \tag{I.5.16}$$

recovering the S, T generators above. At higher rank a determinant-one integer matrix need *not* be symplectic, so $\det M = 1$ alone does not preserve all electric-magnetic pairings. The full condition $M^T J M = J$ retains the information lost by the determinant. This describes changes of frame for the displayed unimodular lattice; a chosen particle or genuine-line lattice can restrict the allowed transformations.

Three dualities that wear the same symbol

The letter "S-duality" and the group $SL(2,\mathbb{Z})$ appear in three distinct places in this chapter, and conflating them is the standing error of the section.

1. *Low-energy electromagnetic duality* on the abelian Coulomb branch (§I.5.2): the group $Sp(2r,\mathbb{Z})$ acting on the lattice $\Gamma_{\rm em}$ and the period vector Π, $SL(2,\mathbb{Z})$ at rank one.
2. *Monodromy* of the period vector around singular loci of the Coulomb branch (§I.5.4): a specific *subgroup* of the low-energy duality group, generated by the degenerations of the Seiberg–Witten geometry (for pure $SU(2)$, the M_m, M_d, M_∞ below).
3. *Exact S-duality* of a full superconformal theory (§I.5.7): a duality of the *interacting* theory at its marginal coupling, e.g. $SU(2)$ with $N_f = 4$, carrying extra flavor data (Spin(8) triality) and global-form caveats.

The first is a property of the Cartan theory, the second is a property of the curve, and the third is a property of the SCFT. They are related but not the same.

This layer comes first because it gives the curve's monodromies a home: the Seiberg–Witten solution attaches to each vacuum a coupling τ and a charge vector, and carrying the vacuum around a singular point acts on the pair by an $Sp(2r,\mathbb{Z})$ matrix (at rank one, $SL(2,\mathbb{Z})$). Those monodromy matrices are exactly the electromagnetic-duality transformations of this section.

I.5.3 The Coulomb branch and rigid special geometry

The Coulomb branch is the family of vacua on which the adjoint scalars are turned on. Their VEVs break the gauge group to its Cartan $U(1)^r$, so the branch has complex dimension the rank,

$$\dim_{\mathbb{C}} \mathcal{M}_C \;=\; r. \tag{I.5.17}$$

It is parametrized not by the gauge-dependent bare VEVs but by the gauge-invariant Casimirs built from ϕ, which are gauge-invariant coordinates on the branch. They differ from the local special coordinates a^I. For $SU(2)$ there is one such invariant,

$$u \;=\; \langle \operatorname{Tr} \phi^2 \rangle, \tag{I.5.18}$$

and the Coulomb branch is the complex u-plane. Calling it "the space of scalar VEVs" is imprecise: the bare VEV $\langle\phi\rangle$ is gauge-dependent, while the branch is coordinatized by the gauge-invariant Casimirs and, locally, by the special coordinates a, a_D the prepotential below builds. The Coulomb branch is not a generic space of scalar expectation values; it carries a rigid local special geometry, the data Seiberg–Witten theory solves.

The two-derivative vector-multiplet Lagrangian on the smooth Coulomb branch is fixed by a single holomorphic function, the *prepotential* $\mathcal{F}(a)$, where a^I are the special coordinates of the r abelian vector multiplets. From $\mathcal{F}$ one builds the dual special coordinates, the low-energy coupling matrix, and the metric,

$$a_{D,I} \;=\; \frac{\partial \mathcal{F}}{\partial a^I}, \qquad \tau_{IJ} \;=\; \frac{\partial^2 \mathcal{F}}{\partial a^I\, \partial a^J} \;=\; \frac{\partial a_{D,I}}{\partial a^J}, \qquad ds^2 \;=\; \mathrm{Im}\, \tau_{IJ}\, da^I\, d\bar{a}^J. \quad \text{(I.5.19)}$$

The second derivative is the period matrix τ_{IJ}, the rank-r coupling whose rank-one entry is the τ of (I.5.7), now holomorphic in the position on the branch. The two ways of writing it agree by construction. The metric is $\mathrm{Im}\, \tau_{IJ}$, its positivity the positivity of the gauge kinetic term. This is what *rigid special geometry* means: one holomorphic prepotential controls the special coordinates, their duals, the coupling matrix, and the metric at once. The prepotential is only *locally* defined, since a duality transformation changes (a, a_D) and with it $\mathcal{F}$; the periods form a section of a flat symplectic bundle over the smooth branch. Its local components $\Pi = (a_{D,I}, a^I)$ are related by $Sp(2r, \mathbb{Z})$ transformations, and $\mathcal{F}$ is a frame-dependent potential. We now specialize to rank one.

A model makes the structure concrete and checkable. At weak coupling the one-loop running of the $SU(2)$ theory gives the prepotential

$$\mathcal{F}(a) \;=\; \frac{i}{2\pi}\, a^2\, \ln \frac{a^2}{\Lambda^2}, \quad \text{(I.5.20)}$$

with Λ the dynamical scale and $|a| \gg |\Lambda|$, where instanton corrections are suppressed. Differentiating determines how the magnetic coordinate depends on the electric one,

$$a_D \;=\; \frac{d\mathcal{F}}{da} \;=\; \frac{i}{\pi}\, a \left(\ln \frac{a^2}{\Lambda^2} + 1 \right), \quad \text{(I.5.21)}$$

The derivative of a^2 supplies the logarithmic term, while the derivative of $\ln(a^2/\Lambda^2)$ supplies the $+1$. Both are required for this choice of prepotential. The second derivative is the running coupling,

$$\tau \;=\; \frac{d^2\mathcal{F}}{da^2} \;=\; \frac{da_D}{da} \;=\; \frac{i}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right), \quad \text{(I.5.22)}$$

The first derivative carries one power of mass; the second is dimensionless, as a gauge coupling must be. The logarithm controls the weak-coupling running: its coefficient i/π is the asymptotic-freedom slope fixed by the one-loop beta function, and the metric is positive in the weak-coupling region,

$$\mathrm{Im}\, \tau \;=\; \frac{1}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right) \;>\; 0 \qquad \text{for real } a \gg \Lambda > 0. \quad \text{(I.5.23)}$$

The additive constants ($+1$ in a_D, $+3$ in τ) depend on the definition of Λ and the scheme; the discriminating data are the logarithm and its slope, not the shift. A prepotential giving $\mathrm{Im}\, \tau < 0$ would signal a wrong-sign kinetic term, and the off-by-one-derivative misidentification $\tau = d\mathcal{F}/da$ would also give the wrong mass dimension. The Coulomb

branch is special-Kähler, with an electromagnetic period system; the Higgs branch instead has a hyperkähler metric and no such vector-multiplet period system.

The higher-rank atlas: $SU(N)$ Coulomb branches. The worked theory below is $SU(2)$, but the structure is general. An $SU(N)$ gauge theory breaks on its Coulomb branch to $U(1)^{N-1}$, so its rank is $r = N - 1$. The gauge-invariant coordinates are the Casimirs of the adjoint scalar,

$$u_k \;=\; \langle \operatorname{Tr} \phi^k \rangle, \qquad k = 2, 3, \ldots, N, \qquad \#\{u_k\} \;=\; N - 1 \;=\; r, \tag{I.5.24}$$

their number matching the rank. (These are the $\mathcal{N} = 2$ counterparts of the $\operatorname{Tr} \phi^k$ chiral-ring generators of the $\mathcal{N} = 1$ chapter.) The low-energy theory is an abelian gauge theory of rank r, with a period vector of $2r$ components $(a_{D,I}, a^I)$, a charge lattice $\Gamma_{\text{em}} \cong \mathbb{Z}^{2r}$ carrying the Dirac pairing (I.5.12), and a duality group $Sp(2r, \mathbb{Z})$. For $N = 2$ this is $r = 1$, one coordinate $u = u_2$, and $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$, the case whose curve and monodromies we study below. The higher-rank Seiberg–Witten curves (hyperelliptic of genus r, with their period matrices) are the developed in the class S and geometric-engineering volumes. The rank-r notation above allows their several electric and magnetic periods to be treated together. Table I.5.1 is the counting.

G	rank r	Coulomb coordinates	period-vector length $2r$
$SU(2)$	1	u_2	2
$SU(3)$	2	u_2, u_3	4
$SU(N)$	$N - 1$	$u_2, \ldots, u_N$	$2(N - 1)$

Table I.5.1: The $SU(N)$ Coulomb-branch atlas. The rank is $r = N - 1$, the Casimir coordinates are $u_k = \langle \operatorname{Tr} \phi^k \rangle$ for $k = 2, \ldots, N$, and the duality group is $Sp(2r, \mathbb{Z})$. The chapter solves the $SU(2)$ ($r = 1$) row in full; the rest fix the notation for the higher-rank constructions developed later.

I.5.4 Seiberg–Witten theory and the central charge

The prepotential is holomorphic but not single-valued: around the singular points of the Coulomb branch it has the $SL(2, \mathbb{Z})$ monodromies of §I.5.2. The Seiberg–Witten solution encodes the exact, non-perturbative $\mathcal{F}$ in geometry. Over each point u it places an auxiliary Riemann surface Σ_u, the *Seiberg–Witten curve*, carrying a meromorphic one-form λ_{SW}, the *Seiberg–Witten differential*. The special coordinate and its dual are the periods of λ_{SW} over a symplectic basis (A, B) of one-cycles,

$$a(u) \;=\; \oint_A \lambda_{\text{SW}}, \qquad a_D(u) \;=\; \oint_B \lambda_{\text{SW}}. \tag{I.5.25}$$

The curve is auxiliary: not the spacetime, not the moduli space, but a counting geometry whose periods compute (a, a_D, τ) and whose degenerations locate the massless states.

The BPS central charge follows from the periods. A state of electric and magnetic charges (n_e, n_m) has central charge and BPS mass

$$Z \;=\; a\, n_e \;+\; a_D\, n_m, \tag{I.5.26}$$

$$M \;=\; \sqrt{2}\,|Z| \;=\; \sqrt{2}\,\big|\, a\, n_e + a_D\, n_m \,\big|. \tag{I.5.27}$$

This is the normalization convention used here, taken verbatim from the Seiberg–Witten solution, the form Chapter 1 already used. The factor $\sqrt{2}$ is part of the convention; dropping it to write $M = |Z|$ is wrong in this normalization. The three species of state read off the charges,

$$\text{W-boson } (1,0):\; M = \sqrt{2}\,|a|; \qquad \text{monopole } n_m \neq 0; \qquad \text{dyon } n_e n_m \neq 0. \tag{I.5.28}$$

Where a one-cycle of Σ_u pinches, that is, where two branch points collide and the discriminant vanishes, the corresponding period vanishes and the BPS state of that charge becomes massless. The $SL(2,\mathbb{Z})$ monodromy around that point is the duality transformation the massless state generates.

Pure $SU(2)$, worked end to end. The cleanest instance is $SU(2)$ with one vector multiplet and no hypermultiplets. The Coulomb branch is the u-plane of (I.5.18). The microscopic $U(1)_R$ is broken by the anomaly to $\mathbb{Z}_8$, and since u has R-charge four, a residual $\mathbb{Z}_2$ acts on the branch as $u \mapsto -u$. The Seiberg–Witten curve, in the standard form, is

$$y^2 \;=\; (x^2 - \Lambda^4)\,(x - u), \tag{I.5.29}$$

a family of elliptic curves over the u-plane. The quartic $y_Q^2 = (v^2 - u)^2 - \Lambda^4$ also occurs in constructions of the pure theory, but it is generically not birational to this cubic at the same u, Λ. The distinction can be seen without computing any physical period. Their invariants are

$$j_Q = \frac{64(4u^2 - 3\Lambda^4)^3}{\Lambda^8(u^2 - \Lambda^4)}, \qquad j_C = \frac{64(u^2 + 3\Lambda^4)^3}{\Lambda^4(u^2 - \Lambda^4)^2}. \tag{I.5.30}$$

For example, at $u = 2, \Lambda = 1$ they are $140608/3$ and $21952/9$. Smooth birational genus-one curves have the same j-invariant, so this difference rules out the proposed identification. Instead the map

$$(x, y_C) = (u - v^2,\; ivy_Q) \tag{I.5.31}$$

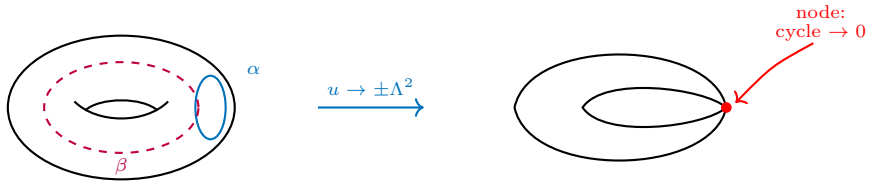
satisfies the cubic equation and identifies (v, y_Q) with $(-v, -y_Q)$. It has degree two on smooth fibers, giving a two-isogeny after choosing origins. Consequently a comparison of physical periods must also translate the differential and the integral charge lattice. Volume VII-A, Chapter 5 develops that translation; we retain the cubic periods and charges throughout this example.

An elliptic curve $y^2 = p(x)$ with p a cubic is a double cover of the x-sphere, branched over the three roots of p together with the point at infinity. Reading off the cubic on the right of (I.5.29), the three finite branch points are

$$x \;=\; +\Lambda^2, \qquad x \;=\; -\Lambda^2, \qquad x \;=\; u, \tag{I.5.32}$$

the first two fixed by the scale and the third moving with the position u on the Coulomb branch. The curve degenerates, and a one-cycle pinches, exactly when the moving root meets a fixed root,

$$u \;=\; \pm\Lambda^2. \tag{I.5.33}$$



(a) generic u: smooth torus (b) $u = \pm\Lambda^2$: the one-cycle pinches

Figure I.5.1: The Seiberg–Witten curve (I.5.29) is a torus fibered over the u-plane. At a generic u (a) it is smooth, with a basis of one-cycles α, β whose periods are $a_D = \oint_\alpha \lambda_{\rm SW}$ and $a = \oint_\beta \lambda_{\rm SW}$. At $u = \pm\Lambda^2$ (b) two branch points collide and a one-cycle pinches: the period of the vanishing cycle goes to zero, so the BPS state with that charge becomes massless. The vanishing cycle is a different one-cycle at each point, the monopole cycle (period a_D) at $+\Lambda^2$ and the dyon cycle (period $-a + a_D$) at $-\Lambda^2$. The curve is an auxiliary Riemann surface, not the spacetime geometry.

The invariant statement is that the discriminant of the cubic in x vanishes. For roots r_i it is $\prod_{i<j}(r_i - r_j)^2$. Here the three differences are $2\Lambda^2$, $u - \Lambda^2$ and $u + \Lambda^2$, so, up to a nonzero u-independent factor,

$$\Delta(u) \;\propto\; (u^2 - \Lambda^4)^2 \;=\; (u - \Lambda^2)^2\,(u + \Lambda^2)^2, \tag{I.5.34}$$

a product of *double* zeros at the two finite points

$$u \;=\; +\Lambda^2 \qquad \text{and} \qquad u \;=\; -\Lambda^2, \tag{I.5.35}$$

and at no other finite value, since $\Delta(0) \neq 0$: the origin is not singular. There is also the semiclassical singularity at $u = \infty$, where the theory is weakly coupled. So there are exactly two strong-coupling singularities, not one. The $\mathbb{Z}_2$ symmetry $u \mapsto -u$ relates the two finite points but does not collapse them: the discriminant (I.5.34) is even in u with two roots. The scale-free quartic $y_Q^2 = (v^2 - u)^2$ has identically zero discriminant for every u; it cannot isolate the two quantum singularities. For nonzero Λ the two polynomial discriminants are

$$\operatorname{disc}_v Q = 256\Lambda^8(u^2 - \Lambda^4), \qquad \operatorname{disc}_x C = 4\Lambda^4(u^2 - \Lambda^4)^2. \tag{I.5.36}$$

A nonzero constant factor changes neither vanishing order. The simple quartic zeros and double cubic zeros therefore carry information about the elliptic family and its integral periods; they are not removable normalization choices. The reduced singular locus is nevertheless the same pair of points. In the cubic charge convention, each has one light BPS hypermultiplet, a monopole at $+\Lambda^2$ and a dyon at $-\Lambda^2$. A double discriminant zero does not count two independent light particles.

At each singular point a BPS state becomes massless, and its central charge follows from (I.5.26),

$$\begin{aligned} u = +\Lambda^2 &: \ (n_e, n_m) = (0,1), \quad Z = a_D; \\ u = -\Lambda^2 &: \ (n_e, n_m) = (-1,1), \quad Z = -a + a_D. \end{aligned} \tag{I.5.37}$$

The monopole period $a_D \to 0$ at $+\Lambda^2$ and the dyon combination $-a + a_D \to 0$ at $-\Lambda^2$. The masslessness is the period degeneration, read through the central charge.

Carrying the special coordinates around each singular point produces an $SL(2,\mathbb{Z})$ monodromy. With the monodromies acting on the column vector $(a_D, a)^T$, the three are

$$M_m = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_d = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \qquad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{I.5.38}$$

the monopole monodromy M_m at $u = +\Lambda^2$, the dyon monodromy M_d at $u = -\Lambda^2$, and the semiclassical monodromy M_∞ at $u = \infty$. Each lies in $SL(2,\mathbb{Z})$,

$$\det M_m \ = \ \det M_d \ = \ \det M_\infty \ = \ 1, \tag{I.5.39}$$

as §I.5.2 requires. The consistency condition is that a large loop enclosing both finite singularities is homotopic to the loop at infinity, giving the ordered product $M_\infty = M_m M_d$. On the period column the rightmost matrix M_d acts first; multiplying,

$$M_m\, M_d \ = \ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ = \ \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ = \ M_\infty, \tag{I.5.40}$$

where the lower-left entry is $(-2)(-1) + (1)(-2) = 0$, the relation closes exactly. The order matters: the reversed product is

$$M_d\, M_m \ = \ \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \ = \ \begin{pmatrix} -5 & 2 \\ -8 & 3 \end{pmatrix} \ \neq \ M_\infty. \tag{I.5.41}$$

This product is the backbone of the solution: the strong-coupling physics, two massless states at finite u, is consistent with the weak-coupling one-loop monodromy at infinity.

The matrices act on the period column $(a_D, a)^T$, whose symplectically paired charge column is ordered (n_m, n_e), while the prose labels charges (n_e, n_m) following (I.5.26). For $\Pi' = M\Pi$, a fixed state has $q' = M^{-T} q$ with $q = (n_m, n_e)^T$; then $q'^T \Pi' = q^T \Pi = Z$. The curve alone does not choose all genuine line operators. In particular, the cubic particle labels must not be identified with a primitive homology basis of the quartic without the degree-two translation above.

Table I.5.2 collects the solution. At weak coupling, where u is large, the periods behave as

$$a \ \sim \ \sqrt{2u}, \qquad a_D \ \sim \ \frac{i}{\pi}\, a \, \ln \frac{a^2}{\Lambda^2}, \tag{I.5.42}$$

reproducing the one-loop running and the asymptotic-freedom logarithm of (I.5.20). The two strong-coupling singularities, by contrast, are invisible to perturbation theory. They are the content of the exact solution.

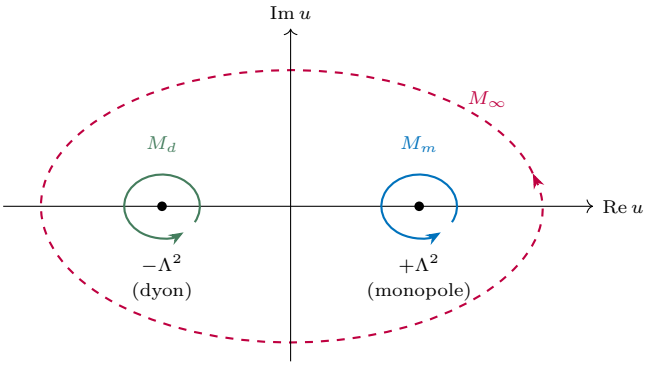


Figure I.5.2: The Coulomb-branch u-plane with its three singularities: the monopole point $u = +\Lambda^2$ (monodromy M_m), the dyon point $u = -\Lambda^2$ (monodromy M_d), and the semiclassical point $u = \infty$ (monodromy M_∞). A large loop enclosing both finite singularities is homotopic to the loop at infinity, so the monodromies obey the ordered product $M_\infty = M_m M_d$ (I.5.40), the consistency condition that ties the strong-coupling physics to the one-loop running at weak coupling.

Local frames: the light monopole and dyon. The weak-coupling variables (a, τ) are not the right description near the singular points. At $u = +\Lambda^2$ the period a_D vanishes, the monopole becomes massless, and the correct low-energy variable is the dual coordinate with small magnetic gauge coupling g_D (and hence large $\operatorname{Im}\tau_D$),

$$\tau_D \;=\; -\frac{1}{\tau}. \tag{I.5.43}$$

In that frame the light monopole is an ordinary hypermultiplet $(M, \widetilde{M})$ of the dual $U(1)$, coupled by the $\mathcal{N} = 2$ vertex of (I.5.4) with the dual photon supermultiplet A_D,

$$W_{\text{local}} \;=\; \sqrt{2}\, A_D\, M\, \widetilde{M}, \tag{I.5.44}$$

a weakly coupled magnetic $\mathcal{N} = 2$ QED whose electron is the monopole. Near $u = -\Lambda^2$ the same structure holds in the dyon frame, reached by a different $SL(2,\mathbb{Z})$ transformation: the light field is the dyon $(-1, 1)$, and the local superpotential is (I.5.44) with

$$A_D \;\to\; A_D - A. \tag{I.5.45}$$

What is strongly coupled in one description is weakly coupled in another, the content of §I.5.2 made local.

This local frame is also the precise seam to the $4d$ $\mathcal{N} = 1$ chapter. Deform by an adjoint mass, written in the low-energy variables near the monopole point,

$$\Delta W \;=\; m\, u \;=\; m\, u(a_D). \tag{I.5.46}$$

This breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ and lifts the Coulomb branch. The condensation is a short F-term computation, worth doing rather than asserting. The combined superpotential

point	massless state (n_e, n_m)	central charge	monodromy on $(a_D, a)^T$
$u = +\Lambda^2$	monopole $(0,1)$	$Z = a_D$	$M_m = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)$
$u = -\Lambda^2$	dyon $(-1,1)$	$Z = -a + a_D$	$M_d = \left(\begin{smallmatrix} -1 & 2 \\ -2 & 3 \end{smallmatrix}\right)$
$u = \infty$	semiclassical	one-loop a_D	$M_\infty = \left(\begin{smallmatrix} -1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$

Table I.5.2: The pure $SU(2)$ Seiberg–Witten solution from the curve (I.5.29). The two finite singularities sit at $u = \pm\Lambda^2$, each carrying one massless BPS dyon, and the semiclassical point at $u = \infty$ closes the consistency relation $M_\infty = M_m M_d$ (I.5.40). The BPS mass at each is $M = \sqrt{2}\,|Z|$ from (I.5.27).

in the local monopole variables $(A_D, M, \widetilde{M})$ is

$$W \;=\; \sqrt{2}\, A_D\, M\, \widetilde{M} \;+\; m\, u(A_D), \tag{I.5.47}$$

and its F-terms are

$$\frac{\partial W}{\partial M} \;=\; \sqrt{2}\, A_D\, \widetilde{M}, \qquad \frac{\partial W}{\partial \widetilde{M}} \;=\; \sqrt{2}\, A_D\, M, \qquad \frac{\partial W}{\partial A_D} \;=\; \sqrt{2}\, M\widetilde{M} \;+\; m\, u'(A_D). \tag{I.5.48}$$

The first two vanish, in a vacuum with $M, \widetilde{M} \neq 0$, only at $A_D = 0$. Setting the third to zero there gives

$$\langle M\widetilde{M} \rangle \;=\; -\frac{m}{\sqrt{2}}\, u'(A_D)\big|_{A_D=0} \;\neq\; 0, \tag{I.5.49}$$

nonzero because u' does not vanish at the monopole point: the monopole condenses. This is the dual Meissner effect, the mechanism of confinement, and it occurs because the adjoint mass breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$. The condensed vacuum is one of the two $\mathcal{N} = 1$ vacua of pure $SU(2)$ super Yang–Mills (one of N_c in the $SU(N_c)$ analogue), the material of Chapter 3. The $\mathcal{N} = 2$ scale Λ and the $\mathcal{N} = 1$ gaugino condensate are separate order parameters, tied only by holomorphic scale matching across the decoupling; we recall the condensate from Chapter 3 rather than conflate the two.

The scale Λ. The scale Λ in the curve (I.5.29) is not put in by hand. It is the $\mathcal{N} = 2$ dynamical scale, generated by dimensional transmutation from the asymptotic freedom of the $\mathcal{N} = 2$ gauge coupling, just as $\Lambda_{\rm QCD}$ is. The connection to the $\mathcal{N} = 1$ condensate appears only after the deformation, through scale matching; the condensate is not the origin of Λ.

The matching is one line of holomorphy. Give the adjoint chiral a mass m and integrate it out. Above m the theory is $\mathcal{N} = 2$; below m it is pure $\mathcal{N} = 1$ super Yang–Mills. The one-loop coefficients and their jump are

$$b_0^{\mathcal{N}=2} \;=\; 2N_c, \qquad b_0^{\mathcal{N}=1} \;=\; 3N_c, \qquad \Delta b_0 \;=\; N_c. \tag{I.5.50}$$

Holomorphic threshold matching at $\mu = m$ equates the two running scales, $\Lambda_{\mathcal{N}=1}^{b_0^{\mathcal{N}=1}} = m^{\,\Delta b_0}\, \Lambda_{\mathcal{N}=2}^{b_0^{\mathcal{N}=2}}$, that is

$$\Lambda_{\mathcal{N}=1}^{3N_c} \;=\; m^{\,N_c}\, \Lambda_{\mathcal{N}=2}^{2N_c}. \tag{I.5.51}$$

The power of m is exactly the jump in the one-loop coefficient, not a free choice. For the worked $SU(2)$ case this reads

$$\Lambda_{\mathcal{N}=1}^{6} \;=\; m^2\,\Lambda_{\mathcal{N}=2}^{4}. \tag{I.5.52}$$

This is the cleanest bridge between this chapter and the $\mathcal{N}=1$ chapter: the $\mathcal{N}=1$ gaugino condensate $\langle\lambda\lambda\rangle \sim \Lambda_{\mathcal{N}=1}^{3}$ of the deformed theory is tied to the $\mathcal{N}=2$ scale only through (I.5.51), with the mass m carrying the threshold. The dimensions agree: the right-hand side has mass dimension $N_c + 2N_c = 3N_c$.

I.5.5 Matter, decoupling, and BPS wall crossing

Pure $SU(2)$ is the cleanest case but not the generic one. Adding N_f hypermultiplets in the fundamental keeps the theory $\mathcal{N}=2$ and enriches both its renormalization-group behavior and its BPS spectrum. Flavor masses contribute to the central charge and create thresholds in the running coupling. Changing the vacuum can also change which charged states remain bound.

The matter central charge and flavor decoupling. With N_f hypers of masses m_f, the BPS central charge gains flavor terms,

$$Z_\gamma \;=\; n_e\,a \;+\; n_m\,a_D \;+\; \sum_{f=1}^{N_f} s_f\,m_f, \tag{I.5.53}$$

Here m_f denotes the coefficient of the chosen flavor charge s_f in Z. Its normalization must be fixed with the flavor lattice and the differential residues; it is not automatically the superpotential parameter in (I.5.4). For an elementary hyper with superpotential mass m at $\Phi = 0$, $M = |m|$ and $M = \sqrt{2}|Z|$ require $|s_f m_f| = |m|/\sqrt{2}$. Magnetic sectors may carry flavor weights that are fractional in an elementary-quark basis. The central charge is linear in the charges (n_e, n_m, s_f), so a state's mass tracks its full charge vector. A flavor-charged state can thus remain massive even when its electromagnetic contribution vanishes. The masses also tie the scales of consecutive theories. Integrating out the heaviest hyper, by sending one mass $m \to \infty$ holding the physics below it fixed, matches the N_f scale to the $N_f - 1$ scale through

$$\Lambda_{N_f-1}^{4-(N_f-1)} \;=\; m\,\Lambda_{N_f}^{4-N_f}, \tag{I.5.54}$$

for $1 \le N_f \le 3$, up to the scale normalization. The exponent is the one-loop coefficient $b_0 = 4 - N_f$; it increases by one when a full hypermultiplet is removed. At the conformal endpoint $N_f = 4$ there is no dynamical scale Λ_4. Its place is taken by a dimensionless holomorphic coupling q_4, in a weak-coupling coordinate normalized so that $\Lambda_3 = m_4 q_4$ in the decoupling limit. Keeping Λ_3 finite while sending m_4 to infinity requires $q_4 \to 0$.

Worked decoupling: $N_f = 4 \to 3 \to 2$ with dimensions checked

Run the chain with mass dimensions tracked line by line. The RG-invariant holomorphic combination of the $SU(2)$ theory with N_f flavors is the power $\Lambda_{N_f}^{b_0}$ with $b_0 = 4 - N_f$ (the scale Λ_{N_f} itself is a mass, of dimension one); it is this holomorphic power we track.

$$N_f = 4: \quad [q_4] = 0, \qquad \text{dimensionless (the marginal coupling);} \tag{I.5.55}$$

$$N_f = 3: \quad \Lambda_3^1 = m_4\, q_4, \qquad [\,\cdot\,] = 1 = 1 + 0\ \checkmark; \tag{I.5.56}$$

$$N_f = 2: \quad \Lambda_2^2 = m_3\, \Lambda_3^1 = m_3\, m_4\, q_4, \qquad [\,\cdot\,] = 2 = 1 + 1 + 0\ \checkmark. \tag{I.5.57}$$

Each integration-out multiplies by one power of the decoupled hypermultiplet mass, raising the exponent by one, so the dimension $[\Lambda_{N_f}^{b_0}] = 4 - N_f$ of the holomorphic power is reproduced at every step. The dimensionless q_4 retains the dependence on the marginal coupling that would be lost by replacing it with the literal number one.

The general $\mathcal{N} = 2$ one-loop coefficient. The $SU(2)$ value $b_0 = 4 - N_f$ is a specialization of the general $\mathcal{N} = 2$ one-loop coefficient

$$b_0 \;=\; 2\,h^\vee(G) \;-\; \sum_{\text{hypers } i} T(R_i), \tag{I.5.58}$$

where $h^\vee(G)$ is the dual Coxeter number and $T(R_i)$ the Dynkin index of the i-th hypermultiplet representation, in the normalization where the fundamental of $SU(N)$ has $T = 1$ (so the adjoint has $T = 2h^\vee$). For $SU(2)$, $h^\vee = 2$ and each fundamental hyper has $T = 1$,

$$b_0 \;=\; 2 \cdot 2 \;-\; N_f \;=\; 4 - N_f, \tag{I.5.59}$$

Each full fundamental hypermultiplet therefore lowers b_0 by one. This subtraction explains both the sequence of asymptotically free theories and the conformal endpoint.

The $SU(2)$ flavor window. Reading (I.5.59), the cases $N_f = 1, 2, 3$ are asymptotically free, each generating a scale Λ_{N_f}, carrying its own matter curve, and keeping an N_f-dependent discrete R-symmetry remnant. The case $N_f = 4$ is special: $b_0 = 0$, so the theory is superconformal with no dynamical scale, carrying instead an exactly marginal complexified coupling τ. Here the conformal statement refers to vanishing hypermultiplet masses; nonzero masses are relevant deformations of that theory.

Table I.5.3 is the window map. It is also the bridge to the S-duality of §I.5.7: the marginal coupling of the $N_f = 4$ row is exactly the parameter that S-duality acts on.

BPS spectrum and walls of marginal stability. Chapter 1 warned that BPS shortening is a statement about a fixed multiplet and does not guarantee a state exists everywhere on moduli space; the pure $SU(2)$ solution illustrates why. The mass formula $M = \sqrt{2}\,|Z_\gamma|$ holds at every point, but *which* charges are realized by stable states can change from region to region. The mechanism is a *wall of marginal stability*. A decay $\gamma \to \gamma_1 + \gamma_2$ (charge conserved) costs no energy only when the central charges align in phase,

$$\arg Z_{\gamma_1} \;=\; \arg Z_{\gamma_2} \qquad \Longrightarrow \qquad |Z_\gamma| \;=\; |Z_{\gamma_1}| + |Z_{\gamma_2}|, \tag{I.5.60}$$

N_f	$b_0 = 4 - N_f$	behavior
0	4	asymptotically free; pure $SU(2)$, the worked solution above
1	3	asymptotically free; own scale Λ_1 and matter curve
2	2	asymptotically free; own scale Λ_2 and matter curve
3	1	asymptotically free; own scale Λ_3 and matter curve
4	0	superconformal; exactly marginal τ, no scale, S-duality of §I.5.7

Table I.5.3: The $SU(2)$ flavor window. The one-loop coefficient is $b_0 = 2N_c - N_f = 4 - N_f$. The three asymptotically free cases ($N_f = 1, 2, 3$) each carry their own dynamical scale and Seiberg–Witten curve; the $N_f = 4$ case is the superconformal theory with a marginal coupling and the four-dimensional S-duality of §I.5.7.

the additive saturation of the bound. This is arithmetic, not a postulate. The central charge is linear in the charge, so

$$Z_{\gamma_1+\gamma_2} \;=\; Z_{\gamma_1} + Z_{\gamma_2}, \tag{I.5.61}$$

and the triangle inequality for complex numbers gives

$$|Z_{\gamma_1+\gamma_2}| \;=\; |Z_{\gamma_1} + Z_{\gamma_2}| \;\le\; |Z_{\gamma_1}| + |Z_{\gamma_2}|, \tag{I.5.62}$$

with equality precisely at the same-phase wall (I.5.60). There the decay is kinematically allowed; off it the strict inequality means a state at γ, if present, is lighter than its would-be constituents and protected against two-body decay. Whether it is actually present in a chamber is settled by the wall-crossing data. For nonzero constituent central charges, phase alignment is the kinematic condition for a marginal decay. A bound state can disappear on crossing such a wall; determining whether it does requires the spectrum on the two sides. The clean distinction is between the *protected mass formula*, the same in every chamber, and the *chamber-dependent spectrum*. Pure $SU(2)$ shows both extremes,

$$\text{weak: } \{(1,0),\,(n,1)\} \text{ (infinite tower)}; \qquad \text{strong: } \{(0,1),\,(-1,1)\} \text{ (two states)}. \tag{I.5.63}$$

At weak coupling (large u) the W-boson and the whole dyon tower survive; at strong coupling only the monopole and dyon that go massless at $u = \pm\Lambda^2$ remain, the tower having decayed across the intervening wall. The systematic jump, the Kontsevich–Soibelman formula and spectral-network construction is developed in Part III. The displayed charge sets count particles up to CPT conjugation.

I.5.6 The hyperkähler Higgs branch

The second branch of an $\mathcal{N}=2$ moduli space is the Higgs branch, parametrized by the hypermultiplet expectation values. With eight supercharges this branch is not merely complex. It is *hyperkähler*: it carries three complex structures, and the $SU(2)_R$ of §I.5.1 rotates them into one another. This is a different geometric structure from the special-Kähler geometry of vector multiplets. For the rigid four-dimensional gauge theories considered here, the Higgs-branch metric is protected against gauge-coupling corrections. It is not described by the electromagnetic period system on the Coulomb branch.

Some theories also have *mixed branches*, where both vector-multiplet scalars and hypermultiplet scalars acquire expectation values. Such branches require the vacuum equations to leave both types of coordinates available. The branch nomenclature is Chapter 2's: Coulomb, Higgs, and mixed are frame-dependent labels read from which multiplets carry the local coordinates.

A worked hyperkähler quotient: $U(1)$ with N hypers. The quotient removes gauge-equivalent scalar configurations while imposing the vacuum equations. To see both operations explicitly, take an $\mathcal{N}=2$ gauge theory with a Higgs branch: a $U(1)$ vector multiplet with N hypermultiplets of unit charge. In $\mathcal{N}=1$ language each hyper is a pair $(Q_i, \widetilde{Q}_i)$, $i=1,\dots,N$, with Q_i of charge $+1$ and $\widetilde{Q}_i$ of charge -1, so the Higgs-branch scalars are $4N$ real, the components of $2N$ complex fields. The vacuum equations are the vanishing of the moment maps. The complex moment map is the F-term of the $\mathcal{N}=2$ coupling (I.5.4), and the real moment map is the $U(1)$ D-term with its Fayet–Iliopoulos parameter ζ,

$$\mu_{\mathbb{C}} \;=\; \sum_{i=1}^{N} Q_i\,\widetilde{Q}_i \;=\; 0, \qquad \mu_{\mathbb{R}} \;=\; \sum_{i=1}^{N} \left(|Q_i|^2 - |\widetilde{Q}_i|^2\right) \;-\; \zeta \;=\; 0. \tag{I.5.64}$$

On the smooth locus where the constraints are independent and the gauge action is free, the complex equation $\mu_{\mathbb{C}}=0$ is one complex, hence two real, conditions; the real equation $\mu_{\mathbb{R}}=0$ is one more; and then one quotients by the $U(1)$ gauge action, removing one further real direction. The hyperkähler quotient is $\mathcal{M}_H = \mu^{-1}(0)/U(1)$, and the real dimension count is

$$\dim_{\mathbb{R}} \mathcal{M}_H \;=\; 4N \;-\; \underbrace{2}_{\mu_{\mathbb{C}}} \;-\; \underbrace{1}_{\mu_{\mathbb{R}}} \;-\; \underbrace{1}_{U(1)} \;=\; 4N-4 \;=\; 4(N-1), \tag{I.5.65}$$

a multiple of four, as a hyperkähler space must be, so the quaternionic dimension is

$$\dim_{\mathbb{H}} \mathcal{M}_H \;=\; N-1. \tag{I.5.66}$$

The three real moment-map conditions ($\mu_{\mathbb{C}}$ counting as two) are the hallmark of the hyperkähler quotient: the $SU(2)_R$ that rotates the three complex structures rotates them into one another, and the three FI levels $(\zeta_{\mathbb{R}}, \operatorname{Re}\zeta_{\mathbb{C}}, \operatorname{Im}\zeta_{\mathbb{C}})$ form an $SU(2)_R$ triplet. We have chosen $\zeta_{\mathbb{C}}=0$ and $\zeta_{\mathbb{R}}=\zeta$. The quotient removes a whole quaternionic direction: three real constraints and one gauge orbit. Counting only the scalar fields would miss

both operations. At nonzero FI parameter the general quotient is the cotangent bundle $T^*\mathbb{P}^{N-1}$, of complex dimension $2(N-1)$. For $N=2$ we can determine the defining equation as well as the dimension. Work at zero FI level, where $\mu_{\mathbb{C}} = Q_1\widetilde{Q}_1 + Q_2\widetilde{Q}_2 = 0$, and form the gauge-invariant mesonic coordinates (each is neutral under the $U(1)$, with Q of charge $+1$ and $\widetilde{Q}$ of charge -1)

$$X \;=\; Q_1\widetilde{Q}_2, \qquad Y \;=\; Q_2\widetilde{Q}_1, \qquad Z \;=\; Q_1\widetilde{Q}_1 \;=\; -\,Q_2\widetilde{Q}_2, \tag{I.5.67}$$

the last equality being the constraint $\mu_{\mathbb{C}} = 0$. These three coordinates are not independent. Multiplying,

$$XY \;=\; (Q_1\widetilde{Q}_1)(Q_2\widetilde{Q}_2) \;=\; Z\,(-Z) \;=\; -Z^2, \qquad \text{i.e.} \qquad XY + Z^2 \;=\; 0, \tag{I.5.68}$$

the A_1 relation: the Higgs branch is the hypersurface $XY + Z^2 = 0$ in $\mathbb{C}^3$, a single quadric cone, which is exactly the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold whose resolution is the Eguchi–Hanson / $T^*\mathbb{P}^1$ space. Before imposing the complex moment map, the same expression is $XY + Z^2 = Z\mu_{\mathbb{C}}$, so the constraint is precisely what produces the surface relation. The nonzero real FI level gives a resolution whose invariant coordinates still map to this cone; the invariant equation alone does not describe the exceptional curve. Volume V, Chapter 4 develops the explicit metrics and the general Kronheimer construction.

The dimension counts now distinguish the two branches: the Coulomb branch is special-Kähler of complex dimension the rank (1 for $SU(2)$), the Higgs branch is hyperkähler of quaternionic dimension $N-1$. The $\mathbb{C}^2/\Gamma$ ALE space of the ADE quiver gauge theory is the geometric realization of exactly this quotient, computed in full in Volume V ch. 4.

I.5.7 S-duality and the marginal coupling

Some four-dimensional $\mathcal{N}=2$ theories have an exactly marginal complexified coupling τ, a coupling whose beta function vanishes so that no scale is generated. The strong- and weak-coupling regimes of such a τ are physically equivalent, related by an S-duality group of the four-dimensional theory. The structure generalizes the abelian electromagnetic duality of §I.5.2 from the low-energy Cartan theory to the full interacting theory.

The canonical example is $SU(2)$ with $N_f = 4$, the bottom row of Table I.5.3. Why this theory is marginal is the one-line beta-function check,

$$b_0 \;=\; 2N_c - N_f \;=\; 4 - N_f \;=\; 0 \quad \text{at} \quad N_f = 4, \tag{I.5.69}$$

with $N_c = 2$: the one-loop coefficient vanishes, no scale is generated, and τ is an exactly marginal parameter. The cancellation is exact for the massless $\mathcal{N}=2$ theory. Its S-duality acts on τ in an $SL(2,\mathbb{Z})$-type fashion, accompanied by Spin(8) triality: the four hypermultiplets fill the vector, and the duality permutes it with the two spinors,

$$\text{hypers} \;\in\; \mathbf{8}_v, \qquad \mathbf{8}_v \;\leftrightarrow\; \mathbf{8}_s \;\leftrightarrow\; \mathbf{8}_c \quad (\text{outer } S_3). \tag{I.5.70}$$

The precise group and action depend on the global form, the line operators, and the masses; we record the massless field-theory fact, "$SL(2,\mathbb{Z})$-type on τ with Spin(8)

object	role under triality
the four hypermultiplet masses	transform as Spin(8) weights
$\mathbf{8}_v,\ \mathbf{8}_s,\ \mathbf{8}_c$	permuted by the outer automorphism S_3
line operators / global form	select the exact duality group

Table I.5.4: The $SU(2)$, $N_f = 4$ superconformal theory under Spin(8) triality. The S-duality acts on τ in an $SL(2,\mathbb{Z})$-type fashion while permuting the three eight-dimensional representations by the outer automorphism. The exact duality group depends on the global form, line operators, and masses, so the table records the structure, not a single universal action.

triality," with the global form and line-operator spectrum specified whenever an exact duality group is needed.

Argyres–Seiberg duality is a further four-dimensional $\mathcal{N} = 2$ S-duality, relating two different superconformal descriptions of the same theory, in which a weakly gauged $SU(2)$ acts on a strongly interacting sector. We state these as four-dimensional facts and use them as such.

There is a systematic organization of these four-dimensional $\mathcal{N} = 2$ S-dualities, together with its higher-dimensional origin and the geometric construction of the curves. The class S construction supplies this organization and is developed in a later volume. Here the essential distinction is between changing the electric-magnetic variables of an abelian effective theory and identifying two descriptions of the full SCFT.

I.5.8 Conformal central charges of a Lagrangian theory

The $\mathcal{N} = 1$ chapter made the conformal central charges a and c a serious computation through a-maximization. The $\mathcal{N} = 2$ side has an even cleaner statement for a Lagrangian superconformal theory: with extended supersymmetry the central charges are fixed by the free-field content alone, with no extremization needed. Counting n_v vector multiplets and n_h hypermultiplets (in units where one free hypermultiplet has $n_h = 1$),

$$a \;=\; \frac{5n_v + n_h}{24}, \qquad c \;=\; \frac{2n_v + n_h}{12}. \tag{I.5.71}$$

The formula is the linear sum over the free-field content,

$$\text{vector}: \;\; (a,c) = \left(\tfrac{5}{24}, \tfrac{1}{6}\right), \qquad \text{hyper}: \;\; (a,c) = \left(\tfrac{1}{24}, \tfrac{1}{12}\right). \tag{I.5.72}$$

Two examples make it concrete. First, $SU(2)$ with $N_f = 4$, the marginal theory of the previous section, whose content is

$$n_v \;=\; \dim SU(2) \;=\; 3, \qquad n_h \;=\; 4 \times \dim \square \;=\; 4 \times 2 \;=\; 8, \tag{I.5.73}$$

so that

$$a \;=\; \frac{5\cdot 3+8}{24} \;=\; \frac{23}{24}, \qquad c \;=\; \frac{2\cdot 3+8}{12} \;=\; \frac{7}{6}. \tag{I.5.74}$$

Second, $\mathcal{N}=4$ super Yang–Mills, read as an $\mathcal{N}=2$ vector multiplet plus one adjoint $\mathcal{N}=2$ hypermultiplet, so $n_v = n_h = \dim G$, and

$$a \;=\; c \;=\; \frac{\dim G}{4}. \tag{I.5.75}$$

For $\mathcal{N}=4$ the equality $a=c$ follows from $n_v = n_h$, so the imbalance of vectors and hypers vanishes,

$$a - c \;=\; \frac{5n_v + n_h}{24} - \frac{2(2n_v + n_h)}{24} \;=\; \frac{n_v - n_h}{24} \;=\; 0. \tag{I.5.76}$$

The $SU(2)$, $N_f = 4$ theory is the smallest member of a conformal Lagrangian family: $SU(N)$ with $N_f = 2N$ fundamentals has $b_0 = 2N - N_f = 0$, marginal for every N, with content

$$n_v \;=\; N^2 - 1, \qquad n_h \;=\; 2N\cdot N \;=\; 2N^2, \tag{I.5.77}$$

giving

$$a \;=\; \frac{5(N^2-1)+2N^2}{24} \;=\; \frac{7N^2-5}{24}, \qquad c \;=\; \frac{2(N^2-1)+2N^2}{12} \;=\; \frac{2N^2-1}{6}, \tag{I.5.78}$$

which at $N=2$ returns $(23/24, 7/6)$, the numbers just computed. Table I.5.5 collects the cases. In particular, the offset -5 in a comes from the missing trace generator of $SU(N)$: replacing N^2-1 by N^2 would change the anomaly.

theory	n_v	n_h	a	c
free vector	1	0	5/24	1/6
free hypermultiplet	0	1	1/24	1/12
$SU(2)$, $N_f = 4$	3	8	23/24	7/6
$\mathcal{N}=4$ SYM, gauge G	$\dim G$	$\dim G$	$\dim G/4$	$\dim G/4$
$SU(N)$, $N_f = 2N$	N^2-1	$2N^2$	$(7N^2-5)/24$	$(2N^2-1)/6$

Table I.5.5: Conformal central charges of Lagrangian $\mathcal{N}=2$ theories from $a = (5n_v + n_h)/24$, $c = (2n_v + n_h)/12$. The $SU(N)$, $N_f = 2N$ row is the conformal family whose $N=2$ member is the $SU(2)$, $N_f = 4$ theory; the $\mathcal{N}=4$ row has $a=c$ because $n_v = n_h$. The equality in the $\mathcal{N}=4$ row follows directly from equal vector and hypermultiplet dimensions.

For a non-Lagrangian theory there are no free fields to count, and the central charges must be determined by other means; the Argyres–Douglas example below is such a case. We determine one combination of its anomalies from its Coulomb dimension; determining them separately requires further information.

There is one more relation, and it is the reason the central charges belong in the same chapter as the Coulomb branch. For the theories considered here, with freely generated Coulomb coordinates and the usual low-energy measure relation, the Shapere–Tachikawa formula determines the combination $2a-c$ from the Coulomb-branch operator dimensions $\{\Delta_i\}$,

$$2a - c \;=\; \frac{1}{4}\sum_i \big(2\Delta_i - 1\big). \tag{I.5.79}$$

Check it on $SU(2)$ with $N_f = 4$, where the single Coulomb coordinate u has $\Delta = 2$: both sides give $3/4$,

$$\frac{1}{4}\big(2\cdot 2 - 1\big) \;=\; \frac{3}{4}, \qquad 2a - c \;=\; 2\cdot\frac{23}{24} - \frac{7}{6} \;=\; \frac{46-28}{24} \;=\; \frac{3}{4}. \tag{I.5.80}$$

The relation earns its keep on the non-Lagrangian side. The rank-one Argyres–Douglas theory of the next section has the fractional Coulomb dimension $\Delta(u) = 6/5$, so even though its a and c separately have not been determined here,

$$2a - c \;=\; \frac{1}{4}\Big(2\cdot\frac{6}{5} - 1\Big) \;=\; \frac{7}{20} \tag{I.5.81}$$

is fixed by the Coulomb-branch geometry alone. Thus one Coulomb dimension determines one combination of anomalies. A second piece of information would be needed to separate a and c.

I.5.9 Argyres–Douglas fixed points

A sharper phenomenon occurs at special points of the Coulomb branch. The pure $SU(2)$ solution had its two massless states, the monopole $(0,1)$ and the dyon $(-1,1)$, at *different* points of the u-plane. In richer theories two BPS states with charges (n_e, n_m) and (n'_e, n'_m) can become massless at the *same* point, and if their Dirac pairing is nonzero,

$$n_e\, n'_m \;-\; n_m\, n'_e \;\neq\; 0, \tag{I.5.82}$$

they are mutually non-local. A monopole and a dyon with nonzero pairing are the canonical example. The nonzero pairing prevents any $SL(2,\mathbb{Z})$ frame from making both charges purely electric. At the isolated Argyres–Douglas degeneration considered below, the resulting light sector has no weakly coupled $\mathcal{N} = 2$ gauge description. Its infrared limit is an interacting superconformal theory. This identification uses the physical Seiberg–Witten solution in addition to the charge-pairing obstruction.

This is the sharpest instance of a Chapter 2 fact: not every supersymmetric field theory has a conventional Lagrangian. An Argyres–Douglas fixed point is a genuine field theory, with well-defined Coulomb-branch and Seiberg–Witten data, flavor symmetries, and central charges; it simply has no weakly coupled gauge description with elementary fields.

Scaling at the simplest Argyres–Douglas point. The qualitative mechanism deserves one explicit computation, and the cleanest one is a scaling analysis of the local

curve. For the simplest rank-one Argyres–Douglas theory, the local Seiberg–Witten geometry is

$$x^2 \;=\; z^3 \;+\; u, \qquad \lambda_{\mathrm{SW}} \;=\; x\,dz, \tag{I.5.83}$$

where u is the Coulomb-branch expectation value and λ_{SW} is the Seiberg–Witten differential. The dimensions of the geometric variables are fixed by two inputs. First, the Seiberg–Witten differential has mass dimension one, because its period $a = \oint \lambda_{\mathrm{SW}}$ is the special coordinate and carries dimension of mass, so

$$[\lambda_{\mathrm{SW}}] \;=\; [x] + [z] \;=\; 1. \tag{I.5.84}$$

Second, the curve $x^2 = z^3$ must be scale-covariant, which forces

$$2[x] \;=\; 3[z]. \tag{I.5.85}$$

Solving (I.5.84)–(I.5.85) gives

$$[z] \;=\; \frac{2}{5}, \qquad [x] \;=\; \frac{3}{5}, \tag{I.5.86}$$

and the dimension of the Coulomb-branch operator u follows from $[u] = [x^2] = 2[x]$ (equivalently $[u] = [z^3] = 3[z]$, consistently),

$$\Delta(u) \;=\; 2[x] \;=\; \frac{6}{5}. \tag{I.5.87}$$

The number is the point. A Coulomb-branch dimension of $6/5$ is *fractional*. A free or weakly coupled gauge theory has integer Coulomb dimensions,

$$\Delta(\mathrm{Tr}\,\phi^k) \;=\; k \;\in\; \mathbb{Z}, \tag{I.5.88}$$

so a fractional dimension is the signature of a genuinely interacting, non-Lagrangian fixed point. The differential matters as much as the equation of the curve: assigning dimension one to dz alone would give $[u] = 3$, a different result. The physical normalization $[\lambda_{\mathrm{SW}}] = 1$ selects the fractional solution.

For this simplest (rank-one) Argyres–Douglas point the chapter can pin only its rank one, its computed dimension $\Delta(u) = 6/5$, the absence of a weakly coupled Lagrangian, and the mutual non-locality of its light states. Its central charges a and c are known from anomaly arguments; their individual determination is beyond the calculation here. The combination $2a - c = 7/20$ already follows from (I.5.81).

An $\mathcal{N}=2$ SCFT data card

A non-Lagrangian $\mathcal{N}=2$ superconformal theory has no weakly coupled fields to list, but it still carries a well-defined set of protected data. The disciplined way to read such a theory is to record:

- the *Coulomb-branch operator dimensions* Δ_i (e.g. $\Delta(u)=6/5$ for the example above), which need not be integers;
- the *Higgs branch* as a hyperkähler variety and the *global (flavor) symmetry*;
- the *flavor central charges* k_F, the coefficient of the flavor-current two-point function;
- the *conformal central charges* a and c;
- the *BPS charge lattice* $\Gamma_{\rm em}$ with its Dirac pairing (I.5.12) and any defect/line-operator data;
- the *Seiberg–Witten curve and differential* (I.5.83), or a class S construction, when one is available.

These quantities let us compare theories without listing elementary fields. Their systematic determination for non-Lagrangian theories is developed in Part III.

The local curve makes the obstruction to a weakly coupled gauge description quantitative: its mass-one differential forces a Coulomb operator of dimension 6/5. The construction of more general such fixed points belongs to the class S development.

Exit checklist

After this chapter the reader can

1. read a $4d$ $\mathcal{N}=2$ theory in $\mathcal{N}=1$ language: a vector multiplet as an $\mathcal{N}=1$ vector plus an adjoint chiral, a hypermultiplet as two conjugate $\mathcal{N}=1$ chirals, with R-symmetry $SU(2)_R\times U(1)_R$ and the canonical coupling $W_{\mathcal{N}=2}=\sqrt{2}\,\widetilde{Q}\Phi Q+m\widetilde{Q}Q$;

2. set up the rank-r charge lattice with its Dirac pairing $\langle\gamma,\gamma'\rangle$, write the central charge $Z_\gamma=n_{e,I}a^I+n_m^I a_{D,I}$, name the duality group $Sp(2r,\mathbb{Z})$ (with $SL(2,\mathbb{Z})=Sp(2,\mathbb{Z})$ at rank one), and distinguish low-energy duality, period monodromy, and exact SCFT S-duality;

3. separate the Coulomb branch (special-Kähler, with prepotential $\mathcal{F}$, special coordinates $a_{D,I}=\partial\mathcal{F}/\partial a^I$, coupling $\tau_{IJ}=\partial^2\mathcal{F}/\partial a^I\partial a^J$, metric $\operatorname{Im}\tau$) from the Higgs branch (hyperkähler), and differentiate the one-loop $\mathcal{F}$ to $a_D=(i/\pi)a(\ln+1)$, $\tau=(i/\pi)(\ln+3)$;

4. write the Seiberg–Witten periods $a=\oint_A\lambda_{\rm SW}$, $a_D=\oint_B\lambda_{\rm SW}$ and the central charge $Z=a\,n_e+a_D\,n_m$ with $M=\sqrt{2}\,|Z|$;

5. run the pure $SU(2)$ solution: the curve, the branch points, the discriminant $\propto(u^2-\Lambda^4)^2$, the singularities at $u=\pm\Lambda^2$, the massless monopole and dyon, the ordered monodromy product $M_\infty=M_mM_d$, and the local monopole/dyon frames $W_{\rm local}=\sqrt{2}A_DM\widetilde{M}$;

6. add matter: the central charge $Z = n_e a + n_m a_D + \sum_f s_f m_f$, the flavor-decoupling relation $\Lambda_{N_f-1}^{4-(N_f-1)} = m\Lambda_{N_f}^{4-N_f}$, the $4-N_f$ window, and the wall-crossing discipline $\arg Z_{\gamma_1} = \arg Z_{\gamma_2}$;

7. compute a hyperkähler quotient ($U(1)$ with N hypers, $\dim_{\mathbb{H}} = N-1$), state the $N_f = 4$ marginal coupling ($b_0 = 0$) and its Spin(8)-triality S-duality with the global-form caveat;

8. compute the Lagrangian $\mathcal{N} = 2$ central charges from n_v, n_h, recovering $(a, c) = (23/24, 7/6)$ for $SU(2)$, $N_f = 4$ and $a = c = \dim G/4$ for $\mathcal{N} = 4$ SYM, and use $2a - c = \frac{1}{4}\sum_i (2\Delta_i - 1)$ as the interface to non-Lagrangian SCFT data;

9. derive the Argyres–Douglas fractional dimension $\Delta(u) = 6/5$, and read an $\mathcal{N} = 2$ SCFT through its protected data, distinguishing computed quantities from those requiring further physical input.

Status and sources

The central-charge convention is $Z = a\, n_e + a_D\, n_m$ and $M = \sqrt{2}\,|Z|$. The supercharge count and R-symmetry follow Chapter 1.

Scope of the calculations. The algebra assumes the supplied curves, multiplet content, BPS identification and anomaly formulas. The $N_f = 4$ matching relation retains the marginal coupling q_4. Replacing the formal zero power of the scale by one does not check that dependence. The displayed cubic and quartic invariants, degree-two map, contragredient charge action and one-loop derivatives are conditional algebraic calculations. Integrating the given angular integrand determines the leading electric normalization. It does not select a microscopic cycle or compute the magnetic period. The magnetic logarithm follows separately by differentiating the supplied one-loop prepotential and from the period asymptotics of the original solution. These computations do not derive the Seiberg–Witten solution or its BPS spectrum. The physical central-charge coefficients are supported by anomaly arguments [ANOM]. Their rational arithmetic is a separate conditional calculation.

§I.5.1 $\mathcal{N} = 2$ multiplets in $\mathcal{N} = 1$ language. The $\mathcal{N} = 2$ vector = ($\mathcal{N} = 1$ vector) $\oplus$ (adjoint chiral) (I.5.1); the hyper = two conjugate $\mathcal{N} = 1$ chirals (I.5.2); the canonical coupling $W_{\mathcal{N}=2} = \sqrt{2}\,\widetilde{Q}\Phi Q + m\widetilde{Q}Q$ (I.5.4), with the $\sqrt{2}$ the central-charge normalization fixed by the second supersymmetry; the $SU(2)_R \times U(1)_R$ R-symmetry (rank 2, dimension 4, not $SU(3)$). The multiplet representations and coupling are physical inputs. (Seiberg and Witten (1994a); the multiplet decomposition is the standard $\mathcal{N} = 1$ language of Chapter 3, Tachikawa (2014)). Additional calculations include the $8Q$ count read off the Chapter 1 ladder and the rank and dimension of the stated R-symmetry group.

§I.5.2 Electromagnetic duality and the charge lattice. The complexified coupling $\tau = \theta/2\pi + 4\pi i/g^2$ (I.5.7); the $SL(2,\mathbb{Z})$ generators S, T with $S : \tau \mapsto -1/\tau$, $T : \tau \mapsto \tau + 1$ (I.5.8); the 't Hooft–Polyakov monopole as the magnetic soliton; the rank-r charge lattice $\gamma = (n_{e,I}, n_m^I)$ (I.5.11) with the Dirac pairing $\langle \gamma, \gamma' \rangle = \gamma^T J \gamma'$ (I.5.12), the period-vector central charge (I.5.14), and the duality group $Sp(2r,\mathbb{Z})$ (I.5.15) with $SL(2,\mathbb{Z}) = Sp(2,\mathbb{Z})$; the three-dualities box. The matrix and lattice identities are conditional calculations [MATH]. Seiberg and Witten (1994a)

supplies the pure electric/magnetic central charge; Seiberg and Witten (1994b), equation (2.9), supplies its optional flavor-charge term. Additional calculations include $\det S = \det T = 1$, the Möbius actions, $S^2 = -\mathbb{1}$, $(ST)^3 = -\mathbb{1}$; the Dirac pairing antisymmetric and integer, the monodromies and S, T all in $Sp(2,\mathbb{Z})$ ($M^T J M = J$), and a det-1 rank-2 shear that is *not* symplectic.

§I.5.3 The Coulomb branch and rigid special geometry. The Coulomb branch of complex dimension = rank, coordinatized by the gauge-invariant Casimir $u = \langle \operatorname{Tr} \phi^2 \rangle$ (I.5.18) (distinct from local special coordinates); the prepotential $\mathcal{F}$ with $a_{D,I} = \partial \mathcal{F}/\partial a^I$, $\tau_{IJ} = \partial^2 \mathcal{F}/\partial a^I \partial a^J = \partial a_{D,I}/\partial a^J$, metric $\operatorname{Im} \tau_{IJ}\, da^I d\bar{a}^J$ (I.5.19), $\mathcal{F}$ local and Π global; the one-loop $\mathcal{F}$ (I.5.20) differentiated to $a_D = (i/\pi) a(\ln(a^2/\Lambda^2) + 1)$ (I.5.21) and $\tau = (i/\pi)(\ln(a^2/\Lambda^2) + 3)$ (I.5.22) (the log load-bearing, the additive constants convention-dependent). The derivatives of the supplied prepotential are conditional calculations [MATH]. Tachikawa (2014) gives the Coulomb-branch and special-coordinate account. Seiberg and Witten (1994a) gives the local prepotential and period/metric framework. P. C. Argyres and Douglas (1995), equations (2.1) and (2.14)–(2.16), gives $SU(N)$ gauge-invariant coordinates, the higher-rank curve and period construction; the rank arithmetic here does not derive that physical construction. Additional calculations include $a_D = d\mathcal{F}/da$, the two routes to τ agree, $\operatorname{Im} \tau > 0$ in the physical region; the one-loop a_D and τ with the $+1$ fixed by the displayed prepotential and the log slope i/π.

§I.5.4 Seiberg–Witten theory and the central charge. The periods (I.5.25); the central charge (I.5.26) and BPS mass (I.5.27) in the Chapter 1 normalization; the pure $SU(2)$ curve (I.5.29), its branch points (I.5.32), the discriminant $\propto (u^2 - \Lambda^4)^2$ (I.5.34) vanishing (doubly) at $u = \pm\Lambda^2$ (I.5.35), the massless monopole $(0,1)$ and dyon $(-1,1)$, the monodromies (I.5.38) on $(a_D, a)^T$, the ordered product $M_m M_d = M_\infty$ multiplied out (I.5.40) with the reversed order $\neq M_\infty$ (I.5.41) (Table I.5.2); the local monopole/dyon frames $W_{\text{local}} = \sqrt{2} A_D M \widetilde{M}$ (I.5.44) and the $\mathcal{N} = 1$ seam $\Delta W = m\, u(a_D)$ under which the monopole condenses, with the explicit F-term solve $A_D = 0$, $\langle M \widetilde{M} \rangle = -(m/\sqrt{2}) u'(0) \neq 0$ (I.5.47)–(I.5.49); the Λ as the $\mathcal{N} = 2$ transmutation scale, tied to the $\mathcal{N} = 1$ gaugino condensate of Chapter 3 only after an $\mathcal{N} = 1$ mass deformation. The full Seiberg–Witten solution, the curve and the special-geometry derivation of $\mathcal{F}$, is treated in Part III. The discriminant, isogeny and transport calculations are conditional mathematics [MATH]. The discriminant is proportional to $(u^2 - \Lambda^4)^2$, with double zeros at $\pm\Lambda^2$ and no zero at the origin. The ordered monodromy product $M_\infty = M_m M_d$ differs from $M_d M_m$. The anomaly statement [ANOM] concerns one coefficient and uses no independent matching calculation. The normalized electric integral uses the displayed integrand. The magnetic logarithm follows from the supplied prepotential. The physical cycle and BPS identifications require additional input. The cubic premise is from Seiberg and Witten (1994a), and the quartic form is from Witten (1997), equations (2.19)–(2.21). The invariant and degree-two comparison uses the translation derived above. Seiberg and Witten (1994b), Section 11.1, supplies the pure-theory index-two charge convention and a related isogeny. The algebraic degree-two map leaves the physical cycle identification and full genuine-line lattice open. Seiberg and Witten (1994b) gives the matter curves, and N. A. Nekrasov (2003) gives the instanton-counting derivation.

§I.5.5 Matter, decoupling, and BPS wall crossing. The matter central charge $Z_\gamma = n_e a + n_m a_D + \sum_f s_f m_f$ (I.5.53); the flavor-decoupling relation $\Lambda_{N_f-1}^{4-(N_f-1)} = m \Lambda_{N_f}^{4-N_f}$ (I.5.54); the $SU(2)$ window $b_0 = 4 - N_f$ (Table I.5.3); the wall of marginal stability $\arg Z_{\gamma_1} = \arg Z_{\gamma_2}$ (I.5.60), the protected mass formula vs the chamber-dependent spectrum, and the pure-$SU(2)$ weak-coupling tower vs strong-coupling two-state chamber (operating Chapter 1's BPS-stability warning). The Kontsevich–Soibelman / spectral-network machinery is treated in later chapters. The exponent arithmetic and the triangle-equality calculation are conditional [MATH]. The flavor, scale and spectrum assertions, including the q_4 matching relation, are cited physical

inputs. (Seiberg and Witten (1994b) the matter curves; Gaiotto, Moore, et al. (2010) the routed wall-crossing side). Additional calculations include the matter central charge linear with the mass term present; the decoupling exponents $4-(N_f-1)$ vs $4-N_f$ stepping by one; the b_0 window arithmetic.

§I.5.6 The hyperkähler Higgs branch. The Higgs branch as hyperkähler (three complex structures rotated by $SU(2)_R$), distinct from the special-Kähler Coulomb branch; mixed branches; the worked $U(1)$-with-N-hypers quotient with the moment maps (I.5.64), the dimension count $4N-2-1-1=4(N-1)$ (I.5.65), and $\dim_{\mathbb{H}}=N-1$ (I.5.66) ($N=2$ the A_1 / $T^*\mathbb{P}^1$); the explicit $\mathbb{C}^2/\Gamma$ ALE geometry is the engine home Volume V ch. 4, with the hyperkähler-quotient construction routed to later chapters. The quotient count and invariant ring follow mathematically [MATH] under the given assumptions. The physical Higgs-branch and resolution identifications are inputs. (Kronheimer (1989) via Volume V.4). Additional calculations include $\dim_{\mathbb{H}}=N-1$ for $N=1,\dots,5$ and the constraint counting; the Coulomb-branch complex dimension = rank, the Coulomb branch carries a special-Kähler metric.

§I.5.7 S-duality and the marginal coupling. The exactly marginal coupling and its four-dimensional S-duality; the canonical $SU(2)$ with $N_f=4$ ($b_0=2N_c-N_f=0$ (I.5.69), marginal τ, an $SL(2,\mathbb{Z})$-type S-duality plus Spin(8) triality $\mathbf{8}_v\leftrightarrow\mathbf{8}_s\leftrightarrow\mathbf{8}_c$, with the global-form/line-operator/mass caveats stated and routed); Argyres–Seiberg duality, stated. The systematic class S organization and its higher-dimensional origin are routed to later chapters and developed nowhere in the body. The $N_f=4$ triality is from Seiberg and Witten (1994b), Sections 6 and 17. (Seiberg and Witten (1994b) the $N_f=4$ marginal coupling; P. C. Argyres and Seiberg (2007) Argyres–Seiberg duality; Gaiotto (2009) the routed class S program). Additional calculations include $b_0=0$ at $N_f=4$ and $b_0>0$ below it, only $N_f=4$ marginal among $0,\dots,4$.

§I.5.8 Conformal central charges of a Lagrangian theory. The Lagrangian $\mathcal{N}=2$ central charges $a=(5n_v+n_h)/24$, $c=(2n_v+n_h)/12$ (I.5.71) fixed by the free-field content; $SU(2)$ with $N_f=4$ giving $(a,c)=(23/24,7/6)$ (I.5.74); $\mathcal{N}=4$ super Yang–Mills ($n_v=n_h=\dim G$) giving $a=c=\dim G/4$ (I.5.75), the $a=c$ equality following from its equal vector and hypermultiplet dimensions; the $SU(N)$, $N_f=2N$ conformal family $a=(7N^2-5)/24$, $c=(2N^2-1)/6$ (I.5.78) reducing to $(23/24,7/6)$ at $N=2$ (Table I.5.5). Non-Lagrangian central charges (the Argyres–Douglas case) are routed onward, not invented. The anomaly formulas [ANOM] give two coefficients a,c in the Lagrangian cases and the combination $2a-c$ in the Shapere–Tachikawa cases. The numerical substitutions are conditional calculations [MATH]. (P. C. Argyres and Douglas (1995) the AD theory; the Lagrangian central-charge formulas and the $2a-c$ relation are textbook $\mathcal{N}=2$ results, the non-Lagrangian determination routed). Additional calculations include the free vector $(5/24,1/6)$ and hyper $(1/24,1/12)$ values, the $N_f=4$ numbers $(23/24,7/6)$, and the $\mathcal{N}=4$ degeneracy $a=c=\dim G/4$. The Shapere–Tachikawa relation $2a-c=\frac{1}{4}\sum_i(2\Delta_i-1)$ (I.5.79), used here under the freely generated Coulomb-coordinate and low-energy measure assumptions, checked on $SU(2)$, $N_f=4$ ($2a-c=3/4$, matching a,c) and used to pin the Argyres–Douglas combination $2a-c=7/20$ (I.5.81) from $\Delta(u)=6/5$ alone.

§I.5.9 Argyres–Douglas fixed points. Two mutually non-local BPS states (nonzero Dirac pairing (I.5.82)) massless together; no duality frame makes both local, so the IR is an interacting non-Lagrangian $\mathcal{N}=2$ SCFT; the scaling example $x^2=z^3+u$, $\lambda_{\mathrm{SW}}=x\,dz$ (I.5.83) giving $[z]=2/5$, $[x]=3/5$ (I.5.86) and the fractional $\Delta(u)=6/5$ (I.5.87); the $\mathcal{N}=2$ SCFT data card. The pairing and scaling calculations are conditional [MATH]. The existence and interpretation of the Argyres–Douglas theory are physical premises. (P. C. Argyres and

Douglas (1995)). Additional calculations include $[z] = 2/5$, $[x] = 3/5$, $\Delta(u) = 6/5$ from $[\lambda] = 1$ and $2[x] = 3[z]$, $\Delta(u)$ non-integer.

Further reading

The Seiberg–Witten solution has the pedagogical review Lerche (1997) and the lecture notes Tachikawa (2014). Instanton counting and the prepotential are in N. Nekrasov and Okounkov (2006), extended to quivers in N. Nekrasov and Pestun (2023); localization on S^4 in Pestun (2012) with the review Pestun et al. (2017); the AGT correspondence in Alday et al. (2010). BPS spectra and wall-crossing appear in Gaiotto, Moore, et al. (2013) and Gaiotto, Moore, et al. (2012). Argyres–Douglas and other non-Lagrangian fixed points are in P. C. Argyres, Plesser, et al. (1996) and Xie (2013), their central charges in Shapere and Tachikawa (2008), the geometric constraints on Coulomb branches in P. Argyres et al. (2018), and the associated chiral algebra in Beem et al. (2015).

For modern entry points, see the comprehensive guide Akhond et al. (2022) and the Coulomb-branch lectures Martone (2020a). Higher-rank central-charge formulae and special-Kähler stratification are developed in Martone (2020b) and P. C. Argyres and Martone (2020), while the protected chiral-algebra sector is reviewed in Lemos (2020).

References

Akhond, M. et al. (2022). "The Hitchhiker's Guide to 4*d* $\mathcal{N} = 2$ Superconformal Field Theories". In: *SciPost Phys. Lect. Notes* 64, p. 1. DOI: 10.21468/SciPostPhysLectNotes.64. arXiv: 2112.14764 [hep-th].

Alday, L. F., D. Gaiotto, and Y. Tachikawa (2010). "Liouville Correlation Functions from Four-dimensional Gauge Theories". In: *Lett. Math. Phys.* 91, pp. 167–197. DOI: 10.1007/s11005-010-0369-5. arXiv: 0906.3219 [hep-th].

Argyres, P. et al. (2018). "Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part I: physical constraints on relevant deformations". In: *JHEP* 02, p. 001. DOI: 10.1007/JHEP02(2018)001. arXiv: 1505.04814 [hep-th].

Argyres, P. C. and M. R. Douglas (1995). "New phenomena in SU(3) supersymmetric gauge theory". In: *Nucl. Phys. B* 448, pp. 93–126. DOI: 10.1016/0550-3213(95)00281-V. arXiv: hep-th/9505062.

Argyres, P. C. and M. Martone (2020). "Towards a classification of rank r $\mathcal{N} = 2$ SCFTs. Part II. Special Kähler stratification of the Coulomb branch". In: *JHEP* 12, p. 022. DOI: 10.1007/JHEP12(2020)022. arXiv: 2007.00012 [hep-th].

Argyres, P. C., M. R. Plesser, et al. (1996). "New N=2 superconformal field theories in four-dimensions". In: *Nucl. Phys. B* 461, pp. 71–84. DOI: 10.1016/0550-3213(95)00671-0. arXiv: hep-th/9511154.

Argyres, P. C. and N. Seiberg (2007). "S-duality in N=2 supersymmetric gauge theories". In: *JHEP* 12, p. 088. DOI: 10.1088/1126-6708/2007/12/088. arXiv: 0711.0054 [hep-th].

Beem, C. et al. (2015). "Infinite Chiral Symmetry in Four Dimensions". In: *Commun. Math. Phys.* 336.3, pp. 1359–1433. DOI: 10.1007/s00220-014-2272-x. arXiv: 1312.5344 [hep-th].

Gaiotto, D. (2009). *N=2 dualities*. arXiv: 0904.2715 [hep-th].

Gaiotto, D., G. W. Moore, and A. Neitzke (2010). "Four-dimensional wall-crossing via three-dimensional field theory". In: *Commun. Math. Phys.* 299, pp. 163–224. DOI: 10.1007/s00220-010-1071-2. arXiv: 0807.4723 [hep-th].

Gaiotto, D., G. W. Moore, and A. Neitzke (2012). "Wall-Crossing in Coupled 2d-4d Systems". In: *JHEP* 12, p. 082. DOI: 10.1007/JHEP12(2012)082. arXiv: 1103.2598 [hep-th].

Gaiotto, D., G. W. Moore, and A. Neitzke (2013). "Framed BPS States". In: *Adv. Theor. Math. Phys.* 17.2, pp. 241–397. DOI: 10.4310/ATMP.2013.v17.n2.a1. arXiv: 1006.0146 [hep-th].

Kronheimer, P. B. (1989). "The construction of ALE spaces as hyper-Kähler quotients". In: *J. Differential Geom.* 29.3, pp. 665–683.

Lemos, M. (2020). "Lectures on chiral algebras of $\mathcal{N} \geqslant 2$ superconformal field theories". In: arXiv: 2006.13892 [hep-th].

Lerche, W. (1997). "Introduction to Seiberg-Witten theory and its stringy origin". In: *Nucl. Phys. B Proc. Suppl.* 55. Ed. by A. Connes, K. Gawedzki, and J. Zinn-Justin, pp. 83–117. DOI: 10.1016/S0920-5632(97)00073-X. arXiv: hep-th/9611190.

Martone, M. (2020a). "The constraining power of Coulomb Branch Geometry: lectures on Seiberg–Witten theory". In: arXiv: 2006.14038 [hep-th].

Martone, M. (2020b). "Towards the classification of rank-r $\mathcal{N} = 2$ SCFTs. Part I. Twisted partition function and central charge formulae". In: *JHEP* 12, p. 021. DOI: 10.1007/JHEP12(2020)021. arXiv: 2006.16255 [hep-th].

Nekrasov, N. and A. Okounkov (2006). "Seiberg-Witten theory and random partitions". In: *Prog. Math.* 244, pp. 525–596. DOI: 10.1007/0-8176-4467-9_15. arXiv: hep-th/0306238.

Nekrasov, N. and V. Pestun (2023). "Seiberg-Witten Geometry of Four-Dimensional $\mathcal{N} = 2$ Quiver Gauge Theories". In: *SIGMA* 19, p. 047. DOI: 10.3842/SIGMA.2023.047. arXiv: 1211.2240 [hep-th].

Nekrasov, N. A. (2003). "Seiberg-Witten prepotential from instanton counting". In: *Adv. Theor. Math. Phys.* 7.5, pp. 831–864. DOI: 10.4310/ATMP.2003.v7.n5.a4. arXiv: hep-th/0206161.

Pestun, V. (2012). "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops". In: *Commun. Math. Phys.* 313, pp. 71–129. DOI: 10.1007/s00220-012-1485-0. arXiv: 0712.2824 [hep-th].

Pestun, V. et al. (2017). "Localization techniques in quantum field theories". In: *J. Phys. A* 50.44, p. 440301. DOI: 10.1088/1751-8121/aa63c1. arXiv: 1608.02952 [hep-th].

Seiberg, N. and E. Witten (1994a). "Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426. [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

Seiberg, N. and E. Witten (1994b). "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD". In: *Nucl. Phys. B* 431, pp. 484–550. DOI: 10.1016/0550-3213(94)90214-3. arXiv: hep-th/9408099.

Shapere, A. D. and Y. Tachikawa (2008). "Central charges of N=2 superconformal field theories in four dimensions". In: *JHEP* 09, p. 109. DOI: 10.1088/1126-6708/2008/09/109. arXiv: 0804.1957 [hep-th].

Tachikawa, Y. (2014). *N=2 supersymmetric dynamics for pedestrians*. Vol. 890. Lect. Notes Phys. Springer. DOI: 10.1007/978-3-319-08822-8. arXiv: 1312.2684 [hep-th].

Witten, E. (1997). "Solutions of four-dimensional field theories via M-theory". In: *Nucl. Phys. B* 500, pp. 3–42. DOI: 10.1016/S0550-3213(97)00416-1. arXiv: hep-th/9703166.

Xie, D. (2013). "General Argyres-Douglas Theory". In: *JHEP* 01, p. 100. DOI: 10.1007/JHEP01(2013)100. arXiv: 1204.2270 [hep-th].

Chapter I.6

$3d$ supersymmetric field theories

Reader contract. A photon in three dimensions has one propagating polarization. At low energies an abelian photon can therefore be described by a compact scalar, the dual photon. Combined with the scalar in a supersymmetric vector multiplet, it gives a Coulomb branch whose local operators carry magnetic flux. This is why three-dimensional gauge dynamics calls for monopole operators rather than scalar vacuum expectation values alone. The chapter first derives the dual-photon and topological-current normalizations, then computes monopole dimensions in SQED. It next uses the round-sphere free energy to determine a free chiral's R-charge, and follows the effect of a Chern–Simons level on monopole charge and duality. ABJM illustrates the leading $N^{3/2}$ growth of that free energy. With eight supercharges, the same monopoles generate a hyperkähler Coulomb branch; mirror symmetry relates it to a Higgs branch that can be computed by a hyperkähler quotient. Chapters 1 and 2 supply the algebra and multiplet grammar, while Chapters 3 and 5 supply the four-dimensional comparisons. The local calculations here use established field-theory formulas and dualities whose general proofs are identified in the chapter-end apparatus.

What this chapter delivers

The $3d$ $\mathcal{N}=2$ multiplets, the real scalar σ in the vector multiplet, and the dualization of the photon (§I.6.1); the dual photon, the Coulomb branch as a quantum object, and the topological $U(1)_J$ current $j_J = \star F/2\pi$ (§I.6.2); monopole operators, the GNO magnetic charge, flux attachment, and the Borokhov–Kapustin–Wu dimension formula worked on $U(1)$ SQED (§I.6.3); F-maximization, the sphere free energy $F[\Delta]$ and the one-loop function $\ell(z)$, the free-chiral value $\Delta = \frac{1}{2}$, and the F-theorem (§I.6.4); Chern-Simons-matter, the quantized level k, and the parity anomaly $k + \frac{1}{2}T(R) \in \mathbb{Z}$ (§I.6.5); ABJM $U(N)_k \times U(N)_{-k}$, the $\mathcal{N}=6 \to \mathcal{N}=8$ enhancement, and the $N^{3/2}$ free energy (§I.6.6); the Aharony and Giveon–Kutasov dualities, their rank maps, and the S^3 partition-function identity as evidence (§I.6.7); the $\mathcal{N}=4$ enhancement, the twin hyperkähler Coulomb and Higgs branches, the $SU(2)_C \times SU(2)_H$ R-symmetry, and the monopole-generated Coulomb ring $V_+V_- = \Phi^{N_f}$ (§I.6.8); and three-dimensional mirror symmetry as the Coulomb $\leftrightarrow$ Higgs exchange, with the basic $N_f = 1$ pair and the $T[SU(2)]$ self-mirror worked (§I.6.9).

I.6.1 $\mathcal{N}=2$ multiplets and the dual photon

A $3d$ $\mathcal{N}=2$ theory has four real supercharges. This is the same count as $4d$ $\mathcal{N}=1$: dimensional reduction of a four-dimensional $\mathcal{N}=1$ theory on a circle gives a three-dimensional $\mathcal{N}=2$ theory, so the two share their supercharge algebra and their R-symmetry $U(1)_R$. The supercharge table of Chapter 1 records this as the $4Q$ row $4d$ $\mathcal{N}=1$ $=$ $3d$ $\mathcal{N}=2$ $=$ $2d$ $(2,2)$. Four real supercharges is twice the minimal three-dimensional spinor: a minimal $3d$ spinor is two-component real, so $3d$ $\mathcal{N}=1$ has two supercharges and $3d$ $\mathcal{N}=2$ has $2\times 2 = 4$, the four-supercharge workhorse. We take the count and the multiplet grammar as given and build the genuinely new three-dimensional structure on top of it.

The R-symmetry is a single abelian $U(1)_R$, inherited from the four-dimensional $U(1)_R$. As in four dimensions, the fermion in a chiral multiplet has R-charge one less than the scalar, $R_\psi = \Delta - 1$ (with Δ the scalar R-charge), and the gaugino has R-charge $+1$. This shift is the input to the monopole dimension (I.6.18) and to F-maximization (§I.6.4); we recall it, and do not re-derive the multiplet R-charges, which are Chapter 2 grammar.

The two multiplets are the ones fixed in Chapter 2, read in three dimensions.

- *Chiral* Φ: a complex scalar ϕ, a Dirac fermion ψ, and an auxiliary field F.
- *Vector* V: a gauge field A_μ, a gaugino, an auxiliary field D, and one extra real scalar that is the first new fact of the chapter. Reducing a four-dimensional vector multiplet to three dimensions, the gauge field A_M, $M=0,1,2,3$, splits into a three-dimensional gauge field A_μ, $\mu=0,1,2$, and one leftover component A_3. That leftover has no gauge index to carry in three dimensions, so it becomes a *real scalar* σ sitting in the vector multiplet alongside the photon:

$$A_M \xrightarrow{4d\to 3d} \big(A_\mu,\ \sigma\big), \qquad \sigma \equiv A_3. \tag{I.6.1}$$

The extra component supplies a real scalar in the $3d$ $\mathcal{N}=2$ vector multiplet. In the gauge-field normalization of the covariant derivative, $[\sigma]=1$ and $[g^2]=1$; a canonically rescaled free scalar instead has dimension $\frac{1}{2}$. For a $U(1)$ theory a nonzero $\langle\sigma\rangle$ gives a real mass $|q\sigma|$ to a field of charge q, so σ parametrizes a direction of the moduli space of vacua just like a four-dimensional adjoint scalar. But in three dimensions this is only half of the story.

The photon dualizes to a scalar. The second new fact is special to three dimensions. Count the physical polarizations. A free photon in d dimensions carries $d-2$ on-shell states, so in $d=3$ it carries

$$d-2 \;=\; 3-2 \;=\; 1 \tag{I.6.2}$$

single physical polarization: the little group is $SO(1)$, trivial. A free real scalar also carries one state, so the two are candidates to be the same degree of freedom. The change of variables that identifies them uses the low-energy equation of motion, not the Bianchi identity. In the free abelian infrared theory the Maxwell equation

$$d\star F \;=\; 0 \tag{I.6.3}$$

makes $\star F$ closed (the Bianchi identity $dF=0$ is a separate statement, the conservation of the topological current j_J of §I.6.2). On a topologically trivial patch $\star F$ is then exact and can be written as the gradient of a scalar,

$$\star F \;=\; \frac{ig^2}{2\pi}\,d\gamma, \tag{I.6.4}$$

the *dual photon* γ. We use Euclidean signature in this derivation and in the linking-sphere charge below, with orientation $dx^1\wedge dx^2\wedge dx^3>0$ and $\star^2=(-1)^{p(3-p)}=+1$ on p-forms. The factor of i in (I.6.4) belongs to the Euclidean saddle; it is essential for a positive scalar kinetic term. This description assumes an unlifted abelian Maxwell region with $g^2>0$, no effective Chern–Simons term, and the $U(1)$ flux lattice with minimal electric charge one. A massive Maxwell–Chern–Simons photon does not give this massless scalar.

The dualization explains the normalization. In the Euclidean weight e^{-S_E}, treat F as an independent two-form and impose its local Bianchi identity with a multiplier:

$$S_E[F,\gamma]=\frac{1}{2g^2}\int F\wedge\star F+\frac{i}{2\pi}\int\gamma\,dF. \tag{I.6.5}$$

Both integrands are three-forms; the first is the component action $\int d^3x\,F_{\mu\nu}F_{\mu\nu}/(4g^2)$. For variations of compact support, integration by parts turns the second term into $-i\int d\gamma\wedge F/(2\pi)$. Varying γ gives $dF=0$ away from insertions. Varying F instead gives

$$\frac{1}{g^2}\star F=\frac{i}{2\pi}d\gamma,\qquad F=\frac{ig^2}{2\pi}\star d\gamma,\qquad S_E[\gamma]=\frac{g^2}{8\pi^2}\int d\gamma\wedge\star d\gamma. \tag{I.6.6}$$

The Gaussian saddle for F is complex, although its original integration contour is real. Substitution gives $-g^2/(8\pi^2)$ from the Maxwell term and $+g^2/(4\pi^2)$ from the multiplier

term, leaving the positive coefficient displayed above. Thus the photon becomes one scalar with kinetic coefficient $K = g^2/(4\pi^2)$ in $S_E = \frac{K}{2}\int d\gamma \wedge \star d\gamma$.

The global input is distinct from the local equation $dF = 0$. We sum over $U(1)$ bundles with

$$\int_{S^2} F \in 2\pi\mathbb{Z}, \qquad \gamma \sim \gamma + 2\pi. \tag{I.6.7}$$

The compact multiplier is dual to this integer flux lattice; its local equation alone does not quantize the flux. Inserting a monopole of flux $2\pi m$ at x imposes $dF = 2\pi m\delta_x$, where δ_x is the oriented delta three-form. Replacing the constraint term by $i\int \gamma(dF - 2\pi m\delta_x)/(2\pi)$ produces the factor $e^{im\gamma(x)}$ in e^{-S_E}. Integer m makes this factor single-valued under the 2π shift. Conversely, the charge-one sector fixes the period to 2π in this normalization. The expression is schematic in a supersymmetric theory, where scalar factors and dressing also enter the monopole operator. Its shift charge is already visible here.

For now the payoff is a counting statement. Assemble the Coulomb-branch coordinates. Classically a $U(1)$ theory has one real scalar σ; dualizing the photon adds one more, the compact γ:

$$\dim_{\mathbb{R}}\, C^{r=1}_{\mathcal{N}=2} \;=\; \underbrace{1}_{\sigma} \;+\; \underbrace{1}_{\gamma} \;=\; 2, \tag{I.6.8}$$

a two-real-dimensional Coulomb branch, parametrized by the pair (σ, γ). This doubling, one from the reduced gauge-field component and one from the dualized photon, is the seed of everything that follows: it is why the $3d$ $\mathcal{N} = 4$ Coulomb branch will be four-real-dimensional and hyperkähler (§I.6.8), and why its coordinates are quantum monopole operators rather than classical VEVs. The classical vacuum equations (F-flatness $F_i = \partial W/\partial\phi_i = 0$ and $3d$ D-flatness modulo gauge equivalence, Chapter 2 grammar) are unchanged; what is new is that one vacuum coordinate, the dual photon, is absent as an elementary scalar in the original gauge variables and is defined through the flux (I.6.4).

At a generic value of σ, charged matter becomes massive. In a nonabelian theory an adjoint σ can also break the gauge group to its Cartan subgroup; in $U(1)$ it leaves the gauge group intact. A chiral multiplet of gauge charge q has a real mass

$$m_{\text{real}} \;=\; q\,\langle\sigma\rangle, \tag{I.6.9}$$

This is the coupling to the vector-multiplet scalar. At a generic point $\langle\sigma\rangle \neq 0$, a charged field of charge $q \neq 0$ is massive and can be integrated out,

$$\langle\sigma\rangle \neq 0 \quad\Longrightarrow\quad m_{\text{real}} = q\,\langle\sigma\rangle \neq 0 \quad \text{for all charged fields,} \tag{I.6.10}$$

leaving an abelian Maxwell description when the effective Chern–Simons level vanishes and no monopole potential lifts the branch. Semiclassically the Coulomb branch looks like the σ line; the quantum story (§I.6.3) fibers the dual photon over it.

I.6.2 The Coulomb branch and the topological $U(1)_J$

The dual photon is not a spectator. For an ungauged topological current associated with a $U(1)$ factor, its shift symmetry acts on local monopoles. Their expectation values give quantum Coulomb coordinates. Consider the current

$$j_J \;=\; \frac{1}{2\pi}\star F, \qquad \partial_\mu j_J^{\,\mu} \;=\; \frac{1}{2\pi}\star dF \;=\; 0, \tag{I.6.11}$$

conserved not by any equation of motion but by the Bianchi identity $dF = 0$. This is the *topological* symmetry $U(1)_J$; its normalization is fixed by the convention $j_J = \star F/2\pi$ (the current for a $U(N)$ factor is $j = \frac{1}{2\pi}\star \operatorname{Tr} F$). The charge it measures is the total magnetic flux,

$$Q_J \;=\; \int_{S^2} \star j_J \;=\; \frac{1}{2\pi}\int_{S^2} F. \tag{I.6.12}$$

The current j_J is a one-form; $\star j_J = F/(2\pi)$ is the two-form integrated on an outward-oriented linking sphere. Given the chosen flux lattice, its charge follows by division. For one flux quantum $\int_{S^2} F = 2\pi$,

$$Q_J \;=\; \frac{1}{2\pi}\int_{S^2} F \;=\; \frac{2\pi}{2\pi} \;=\; 1, \qquad Q_J(n \text{ quanta}) \;=\; \frac{2\pi n}{2\pi} \;=\; n, \tag{I.6.13}$$

so the basic monopole has topological charge exactly 1 and the charge is linear in the flux. The normalization is load-bearing: a mis-normalized current $j = \star F/4\pi$ would give $Q_J = (2\pi)/(4\pi) = \frac{1}{2}$, the wrong count. The current $j_J = \star F/2\pi$ is the one for which the basic monopole has integer charge 1.

> **Common misconception: the $3d$ Coulomb branch is a classical scalar-VEV space**
>
> It is tempting to picture the $3d$ Coulomb branch as the direct analog of the four-dimensional one, a moduli space of classical adjoint scalar VEVs. It is not. The photon dualizes to a compact scalar, so the branch is parametrized by σ *together with* the dual photon γ, and the operators whose VEVs coordinatize it are monopole operators (§I.6.3), not polynomials in a classical field; in $\mathcal{N} = 4$ its chiral ring is monopole-generated and quantum-corrected (§I.6.8). The adjoint-VEV picture is only the semiclassical approximation, valid far out on σ where the theory is weakly coupled.

In the original gauge variables j_J is conserved by the Bianchi identity. In the dual variables it is the shift Noether current, with the Euclidean Ward normalization made explicit:

$$j_J = \frac{1}{2\pi}\star F = iK\, d\gamma, \qquad K = \frac{g^2}{4\pi^2}, \qquad Q_J = \int_{S^2} \star j_J = m. \tag{I.6.14}$$

Indeed a local shift $\delta\gamma = \alpha(x)$ varies the scalar action by $\delta S_E = K\int d\alpha \wedge \star d\gamma$. With $e^{im\gamma(x)}$ inserted, its Ward identity is $d\star(Kd\gamma) = -im\delta_x$ inside correlators. Multiplication by i gives $d\star j_J = m\delta_x$, so the outward sphere measures charge m. Here $Kd\gamma$ is the real Euclidean variational current, while $j_J = iKd\gamma$ is the current normalized to generate the unitary shift $U(c) = e^{icQ_J}$ and $e^{im\gamma} \mapsto e^{imc}e^{im\gamma}$. This

convention accounts for the Euclidean i; using a real Lorentzian relation without continuing the current would mix conventions.

A magnetic point charge is not winding of γ around a point. A point in three dimensions is linked by S^2, and $\pi_2(S^1) = 0$. The charge is the shift Noether charge of a local insertion, schematically $e^{i\gamma}$ for one flux quantum. Winding of a compact scalar around a linking circle describes a different defect. In the original gauge fields a monopole is a disorder insertion, and its charge is the flux (I.6.12).

Two facts about $U(1)_J$ recur throughout the chapter. First, it can be coupled to a background vector multiplet through a mixed Chern–Simons term. The background scalar supplies a Fayet–Iliopoulos (FI) parameter for the dynamical $U(1)$; the BF term below is its gauge-field component. Thus the FI parameter couples to the topological symmetry,

$$\zeta_{\rm FI} \;\longleftrightarrow\; Q_J, \qquad S_{\rm BF} \;=\; \frac{1}{2\pi}\int A_J \wedge F, \tag{I.6.15}$$

where this BF action is written in Lorentzian signature with weight $e^{iS_{\rm BF}}$. The real-time Chern–Simons actions below use the same convention. The FI interpretation is a fact we use in the mirror dictionary (I.6.76). Second, under mirror symmetry $U(1)_J$ is exchanged with a flavor symmetry of the mirror theory (§I.6.9): the topological current, which has no Lagrangian origin, becomes an ordinary flavor current with one.

Part I. The $3d$ $\mathcal{N} = 2$ language and Chern–Simons matter

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The dual photon gives magnetic flux a local operator. The next step is to compute the dimension of that monopole operator from the charged fields in its flux background. The same charged fields contribute to the sphere free energy that selects the superconformal R-symmetry. A Chern–Simons level then attaches electric charge to the flux, changing which monopoles can appear as gauge-invariant operators and hence changing the duality map. We develop this sequence in $\mathcal{N} = 2$ language, which also describes ABJM before its supersymmetry enhancement. The eight-supercharge theory then reuses the monopole operators to construct its Coulomb branch.

I.6.3 Monopole operators and their dimensions

A $3d$ monopole operator is a local operator, but not one built from the elementary fields. It is a *disorder* operator: to insert it at a point x, one declares that the path integral is taken over field configurations with a prescribed singularity at x. On a small two-sphere S^2 surrounding x, the gauge field is required to carry the flux of a Dirac monopole of magnetic charge m,

$$\frac{1}{2\pi}\int_{S^2} F \;=\; m, \tag{I.6.16}$$

with m valued in the GNO (Goddard–Nuyts–Olive) lattice, the coweight lattice of the gauge group in its declared global form. For $U(1)$ the charge m is an integer; for a nonabelian group it is a cocharacter, an embedding $U(1) \hookrightarrow T$ into the maximal torus, taken modulo the Weyl group. The operator is well defined because the flux (I.6.16) is conserved: inserting it creates one unit (or m units) of the $U(1)_J$ charge (I.6.12), so a bare monopole operator of magnetic charge m carries topological charge $Q_J = m$.

To read off its quantum numbers one uses the state-operator map. In a conformal field theory a local operator at the origin of $\mathbb{R}^3$ corresponds, under the conformal map

$$\mathbb{R}^3 \setminus \{0\} \;\cong\; S^2 \times \mathbb{R}, \qquad \Delta(\mathcal{O}) \;=\; E\big(\text{state on } S^2\big), \tag{I.6.17}$$

to a state on S^2, the radial coordinate becoming Euclidean time, so the operator dimension is the energy of that state. The monopole operator maps to a state of the theory quantized on S^2 *in the background of the magnetic flux* m. At the superconformal point the energy equals the R-charge of the lowest state in the flux sector. The flux deforms the mode expansion of every charged field: a charged fermion in a monopole background develops Landau levels on S^2, and its filled Dirac sea carries a net R-charge. Summing these contributions gives the dimension. The counting is the Borokhov–Kapustin–Wu formula. Written so the matter and the gauge (gaugino) contributions are kept separate,

$$\Delta(m) \;=\; \frac{1}{2} \sum_{\text{chirals } i} \big(1 - \Delta_i\big) \sum_{\rho_i \in R_i} \big|\rho_i(m)\big| \;-\; \sum_{\alpha > 0} \big|\alpha(m)\big|. \tag{I.6.18}$$

Here the sum $\sum_i$ runs over the chiral multiplets; Δ_i is the R-charge of the i-th chiral, so its matter fermion has R-charge $R_\psi = \Delta_i - 1$, and the prefactor $(1 - \Delta_i) = -R_\psi$ is that fermion R-charge with a sign. The inner sum $\sum_{\rho_i}$ runs over the weights ρ_i of the gauge representation R_i, and $\rho_i(m)$ is the pairing of the weight with the magnetic charge. The last sum runs over the positive roots α; it is the gaugino contribution of the vector multiplet, and it is written separately from the matter sum precisely so it is not double-counted with it. The matter contribution raises the dimension and the gaugino contribution lowers it. This is the one canonical dimension formula of the chapter; every special evaluation below is a special case of (I.6.18), never a different convention.

Worked instance: the $U(1)$ SQED monopole

Take $U(1)$ SQED with N_f flavors: N_f hypermultiplets, each of which is two chiral multiplets Q (charge $+1$) and $\widetilde{Q}$ (charge -1), so $2N_f$ chirals of unit charge. Feed this content into (I.6.18) piece by piece. The gauge group $U(1)$ is abelian, so it has *no* positive roots and the gaugino sum drops:

$$U(1) \text{ abelian} \quad\Longrightarrow\quad \sum_{\alpha > 0} \big|\alpha(m)\big| \;=\; 0. \tag{I.6.19}$$

Each of the $2N_f$ chirals carries a single weight of charge ± 1, whose pairing with the magnetic charge is

$$\sum_{\rho_i \in R_i} \big|\rho_i(m)\big| \;=\; |\pm 1 \cdot m| \;=\; |m| \qquad \text{(one weight per chiral)}. \tag{I.6.20}$$

In the $\mathcal{N}=4$ theory the hypermultiplet chirals have R-charge $\Delta_i = \frac{1}{2}$, fixed by the R-symmetry, so every prefactor is $1-\Delta_i = \frac{1}{2}$. Summing the identical contribution of all $2N_f$ chirals,

$$\Delta(m) \;=\; \frac{1}{2}\underbrace{\left(1-\tfrac{1}{2}\right)}_{\text{per chiral}}\underbrace{(2N_f)}_{\#\text{ chirals}}\,|m| \;-\; 0 \;=\; \frac{N_f}{2}\,|m|, \qquad \text{so} \qquad \Delta(1)=\frac{N_f}{2}. \tag{I.6.21}$$

The basic monopole ($m=1$) has dimension $N_f/2$. This is the basic monopole dimension of the chapter, and it is an evaluation of the general formula, not a separate rule. The dimension is genuinely built from the matter: the R-charge enters through the prefactor, so for a generic chiral R-charge Δ_i the same theory gives

$$\Delta(1)\big|_{\Delta_i} \;=\; \frac{1}{2}\,(1-\Delta_i)\,(2N_f) \;=\; N_f\,(1-\Delta_i), \tag{I.6.22}$$

which equals $N_f/2$ only at $\Delta_i=\frac{1}{2}$ (at $\Delta_i=\frac{2}{3}$ it is $N_f/3$ instead). A matter-blind assignment $\Delta=0$ is simply wrong: it ignores the flavor content.

To isolate the gaugino subtraction as a *formal* check (the output is negative, so this is a sign test that exercises the root term, not a physical unitary monopole sector), keep the same two unit-charge chirals ($N_f=1$) but add one positive root of an $SU(2)$-style pairing $\alpha(m)=2m$. The matter sum is unchanged, and the gaugino sum switches on:

$$\Delta_{\text{formal}}(1) \;=\; \underbrace{\tfrac{1}{2}(1-\tfrac{1}{2})(2)|1|}_{\text{matter}=\frac{1}{2}} \;-\; \underbrace{|\alpha(1)|}_{=2} \;=\; \tfrac{1}{2}-2 \;=\; -\tfrac{3}{2}, \tag{I.6.23}$$

lower by exactly 2 than the no-root value $\frac{1}{2}$. The chosen matter weights and root are only an algebraic sign test; they do not specify a consistent $SU(2)$ representation. Folding the gaugino term into the matter sum, or using a fermion R-charge other than Δ_i-1, gives a different (and wrong) dimension. Table I.6.1 collects the SQED evaluation across small N_f.

N_f	chirals	$\Delta(1)=N_f/2$	Coulomb geometry $V_+V_-=\Phi^{N_f}$
1	2	$\frac{1}{2}$	$\mathbb{C}^2$ (smooth)
2	4	1	$\mathbb{C}^2/\mathbb{Z}_2$ (A_1)
3	6	$\frac{3}{2}$	$\mathbb{C}^2/\mathbb{Z}_3$ (A_2)
4	8	2	$\mathbb{C}^2/\mathbb{Z}_4$ (A_3)

Table I.6.1: The basic monopole dimension $\Delta(m=1)=N_f/2$ of $U(1)$ SQED at $\Delta_i=\frac{1}{2}$, from (I.6.18) ($2N_f$ unit-charge chirals, no roots), and the Coulomb-branch geometry $V_+V_-=\Phi^{N_f}$ it seeds (§I.6.8), the A_{N_f-1} singularity.

For general magnetic charge the pairing of each unit-charge weight is $|m|$, so the dimension is linear and doubles from $m=1$ to $m=2$:

$$\Delta(m) \;=\; \frac{N_f}{2}\,|m|, \qquad \Delta(2) \;=\; N_f \;=\; 2\,\Delta(1). \tag{I.6.24}$$

The dimension is a function of the magnetic charge, not a fixed number attached to "the monopole", and the lowest monopole ($m = \pm 1$) generates the Coulomb-branch chiral ring in §I.6.8. When F-maximization (§I.6.4) moves the R-charges off $\frac{1}{2}$, the same formula (I.6.18) responds through the prefactor $(1 - \Delta_i)$; this is how monopoles enter the F-maximization of a gauge theory.

Bare and dressed monopoles. The operator constructed so far is the *bare* monopole: it imposes the flux and nothing else. Acting with the elementary fields in the flux background builds *dressed* monopoles. In $\mathcal{N} = 4$ the dressed tower of the vector-multiplet adjoint scalar Φ,

$$V_m, \quad V_m\,\Phi, \quad V_m\,\Phi^2, \ \ldots \qquad (m = \pm 1), \tag{I.6.25}$$

together with Φ itself generates the Coulomb-branch chiral ring (§I.6.8). This is the precise sense in which the Coulomb branch is "built from monopole operators": its coordinate ring has monopole generators, and the relation $V_+ V_- = \Phi^{N_f}$ fixes a monopole-antimonopole product to a matter-determined power of the adjoint scalar.

Flux attachment. When the gauge field carries a Chern-Simons term of level k (§I.6.5), a monopole of magnetic charge m is no longer gauge-neutral. The Chern-Simons term ties the magnetic flux to electric charge: the Gauss law reads $\frac{k}{2\pi} F_{12} = j^0$, so a unit of flux carries a unit of gauge charge weighted by k. A monopole of charge m therefore acquires a gauge charge

$$q_{\text{gauge}} \;=\; k\,m, \tag{I.6.26}$$

flux attachment (derived from the Gauss law in §I.6.5). Two consequences follow: a bare monopole in a Chern-Simons theory is gauge-charged, so it must be dressed by matter to be gauge-invariant; and flux attachment is why the Giveon–Kutasov dual (§I.6.7) has no monopole singlets while the Aharony dual does.

I.6.4 F-maximization

When several abelian symmetries can mix into the R-symmetry, the superconformal R is a definite linear combination, fixed by an extremization principle. In four dimensions it maximizes the central charge a (Chapter 3); in three dimensions it maximizes the real part of the round-sphere free energy. This is F-maximization, the $3d$ counterpart of the a-maximization in Chapter 3.

The object that is extremized is the free energy of the theory on a round three-sphere. Supersymmetric localization reduces the S^3 partition function to a finite-dimensional matrix integral, a function $Z[\Delta]$ of the trial R-charges Δ_i assigned to the matter. A chiral multiplet of trial R-charge Δ contributes a factor to Z whose logarithm is the one-loop function

$$\ell(z) \;=\; -z\log\bigl(1 - e^{2\pi i z}\bigr) + \frac{i}{2}\Bigl(\pi z^2 + \frac{1}{\pi}\,\mathrm{Li}_2\bigl(e^{2\pi i z}\bigr)\Bigr) - \frac{i\pi}{12}, \tag{I.6.27}$$

evaluated at $z = 1 - \Delta$. The one-loop function has the clean real-axis derivative

$$\ell'(z) \;=\; -\pi z\cot(\pi z), \tag{I.6.28}$$

which is the handle by which the stationarity equations are written. The physical free energy is

$$F[\Delta] \;=\; -\log\bigl|Z[\Delta]\bigr|, \qquad Z_{\text{free chiral}}(\Delta) = \exp \ell(1-\Delta), \tag{I.6.29}$$

and F-maximization identifies the exact superconformal R-charges Δ_i^* at a *local maximum* of $\operatorname{Re} F$ along the allowed mixing directions. For the free chiral below, its one-variable Hessian is strictly negative.

Worked instance: the free chiral

A single free chiral multiplet, with no superpotential and no flavor mixing, has one trial R-charge Δ and free energy $F(\Delta) = -\operatorname{Re}\ell(1-\Delta)$. To extremize it, use the derivative identity (I.6.28). By the chain rule,

$$\frac{\partial F}{\partial \Delta} \;=\; -\operatorname{Re}\Bigl[\ell'(1-\Delta)\cdot(-1)\Bigr] \;=\; \operatorname{Re}\ell'(1-\Delta) \;=\; -\pi(1-\Delta)\cot\bigl(\pi(1-\Delta)\bigr), \tag{I.6.30}$$

using $\ell'(z) = -\pi z\cot(\pi z)$ at $z = 1-\Delta$. The stationarity equation $\partial_\Delta F = 0$ is $\cot(\pi(1-\Delta)) = 0$ (with the prefactor $1-\Delta$ nonzero on $0<\Delta<1$), which is solved by $\pi(1-\Delta) = \pi/2$, that is

$$\Delta^* \;=\; \frac{1}{2}, \qquad \left.\frac{\partial^2 \operatorname{Re} F}{\partial \Delta^2}\right|_{\Delta^*} \;=\; -\frac{\pi^2}{2} \;<\; 0, \tag{I.6.31}$$

the free-field R-charge $\Delta = \frac{1}{2}$, and the Hessian is negative, confirming a maximum. The value $\frac{1}{2}$ is exactly the free-field scaling dimension $(d-2)/2$ of a $3d$ scalar and saturates the $3d$ chiral-primary relation $\Delta = R$ (the three-dimensional analog of the four-dimensional $\Delta = \frac{3}{2}R$, recalled from Chapter 1). For example, direct evaluation gives

$$F(0.4) \;<\; F\bigl(\tfrac{1}{2}\bigr) \;>\; F(0.6)\,: \tag{I.6.32}$$

these nearby trial charges give smaller free energy. Once flavor or monopole mixing is present the extremum moves off $\frac{1}{2}$ to an interacting value determined by the same stationarity equation; such interacting fixed points require the localization calculation of Part III.

The stationarity equation identifies a candidate R-charge; its Hessian distinguishes a maximum from a minimum. Here the negative second derivative $-\pi^2/2$ establishes the local maximum. For a gauge theory, the matter determinant must also be integrated over the vector-multiplet scalar before the same criterion is applied.

Common misconception: F-maximization is a-maximization in three dimensions

The extremization principle is shared, the quantity is not. F-maximization extremizes the round-sphere free energy $F = -\log|Z[\Delta]|$, a *transcendental* function of the trial R-charges built from $\ell(z)$ by localization; it does *not* extremize the four-dimensional fermion-trace cubic $\frac{3}{32}(3\operatorname{Tr}R^3 - \operatorname{Tr}R)$. The superconformal R-charges are a *local maximum* of $\operatorname{Re}F$ (Jafferis), certified by a negative Hessian; F is not a cubic, and its third derivative does not vanish. The F-theorem, $F_{UV} > F_{IR}$, is its monotonicity statement, the $3d$ analog of the a-theorem, and F is a different number from a.

The F-theorem is the monotonicity companion. Under renormalization-group flow the round-sphere free energy decreases,

$$F_{UV} \;>\; F_{IR}, \tag{I.6.33}$$

so F counts degrees of freedom the way a does in four dimensions. A clean way to anchor the theorem is the free energy of the free chiral itself: evaluating (I.6.29) at $\Delta^* = \frac{1}{2}$ gives

$$F_{\text{free chiral}} \;=\; -\operatorname{Re}\ell\big(\tfrac{1}{2}\big)\,, \tag{I.6.34}$$

a fixed, finite real number, the S^3 free energy of one free chiral multiplet (computed numerically from ℓ). A worked monotonicity check follows from giving one of several chirals a real mass and integrating it out: the infrared theory has one fewer massless chiral, so

$$F_{UV} \;=\; n\,F_{\text{free chiral}}, \qquad F_{IR} \;=\; (n{-}1)\,F_{\text{free chiral}}, \qquad F_{UV}{-}F_{IR} \;=\; F_{\text{free chiral}} \;>\; 0, \tag{I.6.35}$$

a strict decrease consistent with (I.6.33). We state the theorem and route its proof (with the localization derivation of $Z[\Delta]$ and the exact value of $F_{\text{free chiral}}$) to Part III.

Worked instance: the structure of SQED F-maximization

The step up to an interacting gauge theory is worth stating, with the scope honest: this chapter runs F-maximization to a number only for the free chiral; the gauge-theory extremum is stated here without a numerical solution. Take $U(1)$ SQED with N_f flavors as an $\mathcal{N} = 2$ theory. The trial R-charge of the $2N_f$ chirals, Δ_Q, is one parameter, but not the whole story: the topological $U(1)_J$ and the flavor $U(1)$ can mix into the R-symmetry, with FI and real-mass parameters coupling to them. The sphere free energy includes an integral over the vector-multiplet scalar σ. At zero Chern–Simons level, a localized integral with conjugate chiral charges takes the form

$$F[\Delta_Q,\zeta] \;=\; -\log\left|\int_{\mathbb{R}} d\sigma\, \exp\big\{N_f\big[\ell(1-\Delta_Q+i\sigma)+\ell(1-\Delta_Q-i\sigma)\big]+2\pi i\zeta\sigma\big\}\right|, \tag{I.6.36}$$

and F-maximization is the same stationarity condition as before, now in several variables,

$$\frac{\partial F}{\partial \Delta_Q} \;=\; 0, \qquad \frac{\partial F}{\partial(\text{mixing})} \;=\; 0, \qquad \operatorname{Hess} F \;\prec\; 0 \text{ (a maximum)}, \tag{I.6.37}$$

which fixes the exact superconformal R when the trial symmetries and convergence conditions are specified. The integral, rather than its exponent at $\sigma = 0$, is the object to extremize; topological mixing changes the background parameters. Its interacting extremum need not equal $\frac{1}{2}$. The full evaluation requires the localization machinery developed in Part III; this chapter delivers the principle, the ℓ-function, and the free-chiral extremum in full.

The extremization family. F-maximization sits in the family Chapter 3 opened, one member per dimension:

$$4d:\ \max_R\, a(R)\ \text{(cubic)}, \quad 3d:\ \max_R\, \operatorname{Re} F[R]\ \text{(transcendental)}, \quad 2d:\ \text{c-extremization}, \tag{I.6.38}$$

the same idea, the exact R fixed by extremizing a trace- or localization-built quantity, realized by a different object in each dimension. Common to all are the discriminating Hessian and a monotonicity theorem (a-theorem, F-theorem); what differs is the quantity, transcendental rather than polynomial in three dimensions. This chapter owns the $3d$ member and signposts the prototype to Chapter 3.

I.6.5 Chern-Simons-matter theories and the parity anomaly

Three dimensions admits a gauge coupling with no four-dimensional analog. A gauge field can carry a Chern-Simons term,

$$S_{\rm CS} \;=\; \frac{k}{4\pi}\int \mathrm{Tr}\Big(A\,dA+\tfrac{2}{3}A^3\Big), \qquad k\in\mathbb{Z} \quad \text{(pure bosonic normalization)}, \tag{I.6.39}$$

with the pure bosonic level k quantized to an integer. The book always quotes the *bare* level k; effective or shifted levels are derived statements, written explicitly when they arise. Quantization of k is a topological requirement. Under a large gauge transformation of winding number $w\in\mathbb{Z}$ the Chern-Simons action shifts by $2\pi k w$, so for a pure non-spin Chern–Simons theory the requirement that the path-integral weight be unchanged reads

$$S_{\rm CS} \;\to\; S_{\rm CS}+2\pi k\,w, \qquad e^{iS_{\rm CS}}\ \text{invariant} \quad\Longleftrightarrow\quad k\in\mathbb{Z}. \tag{I.6.40}$$

The spin theories with fermions used below refine this statement: the bare level can be integer or half-integer, but only subject to the parity-anomaly condition (I.6.43). An arbitrary fraction such as $k=\frac{3}{4}$ is not admissible.

The Chern-Simons term changes the gauge dynamics in two ways we use. First, it gives the photon a topological mass $\sim kg^2$, so a pure Chern-Simons gauge field is gapped rather than dualizing to a massless dual photon; the Coulomb branch is lifted. Second, it attaches flux to charge. Vary (I.6.39) for a $U(1)$ factor together with the matter current: the equation of motion for the time component A_0 reads

$$\frac{k}{2\pi}\,F_{12} \;=\; j^0, \tag{I.6.41}$$

the $3d$ Gauss law with a Chern-Simons source. Integrating over a spatial slice, the left side is $\frac{k}{2\pi}\int F = k\,m$ for a configuration of magnetic charge m, and the right side is the total gauge charge, so a monopole of magnetic charge m carries gauge charge

$$q_{\rm gauge} \;=\; k\,m, \tag{I.6.42}$$

which is (I.6.26), now derived from the Gauss law rather than quoted. The Chern-Simons term thus ties magnetic flux to gauge charge: with $k\neq 0$ a bare monopole is gauge-charged, so it is not a gauge-invariant operator on its own and must be dressed by matter to become one. Depending on the global form of the gauge group and the level, the continuous topological symmetry $U(1)_J$ can then be reduced or mixed with

the gauge data rather than remaining an independent global symmetry. We do not say the Chern-Simons term simply Higgses $U(1)_J$; the precise statement is flux attachment and the dressing it forces. This is the origin of the difference between the Aharony and Giveon–Kutasov dualities in §I.6.7.

The parity anomaly. A single massless Dirac fermion in three dimensions cannot be quantized in a way that preserves both gauge invariance and parity. Regulating the fermion determinant induces a half-integer Chern-Simons term, so a theory with an odd number of charged fermions requires a *half-integer* bare level to be consistent. This is the parity anomaly. For the abelian and fundamental examples used here, the local condition in the stated trace normalization is

$$k + \tfrac{1}{2}\,T(R) \;\in\; \mathbb{Z}, \tag{I.6.43}$$

where $T(R)$ is the index (the Dynkin sum) over the charged fermions of R. For the $U(1)$ or $U(N)$-fundamental example with N_f unit-charge fermions the index is $T(R) = N_f$, so (I.6.43) becomes

$$k + \frac{N_f}{2} \;\in\; \mathbb{Z} \qquad \Longrightarrow \qquad k \in \begin{cases} \mathbb{Z}, & N_f \text{ even}, \\ \tfrac{1}{2} + \mathbb{Z}, & N_f \text{ odd}. \end{cases} \tag{I.6.44}$$

The condition is computed from the index: $T(R) = 3$ forces $k \in \frac{1}{2} + \mathbb{Z}$, $T(R) = 4$ admits $k \in \mathbb{Z}$, and a level outside $\frac{1}{2}\mathbb{Z}$ is inadmissible altogether: for $T(R) = 1$,

$$k = \tfrac{1}{3} \implies k + \tfrac{1}{2} = \tfrac{5}{6} \notin \mathbb{Z}, \tag{I.6.45}$$

so $k = \frac{1}{3}$ is forbidden. The $N_f/2$ form (I.6.44) is the fundamental special case of the general representation-dependent statement (I.6.43); Table I.6.2 tabulates small N_f.

N_f	$T(R) = N_f$	$k + N_f/2 \in \mathbb{Z}$ requires	admissible bare k
1	1	$k \in \frac{1}{2} + \mathbb{Z}$	half-integer
2	2	$k \in \mathbb{Z}$	integer
3	3	$k \in \frac{1}{2} + \mathbb{Z}$	half-integer
4	4	$k \in \mathbb{Z}$	integer

Table I.6.2: The parity anomaly on the $U(1)/U(N)$-fundamental example. The index is $T(R) = N_f$, and the admissible bare Chern-Simons level satisfies $k + N_f/2 \in \mathbb{Z}$: half-integer for odd N_f, integer for even N_f.

I.6.6 ABJM and the $N^{3/2}$ free energy

The canonical Chern-Simons-matter theory is ABJM (Aharony–Bergman–Jafferis–Maldacena). It is a

$$U(N)_k \times U(N)_{-k} \tag{I.6.46}$$

gauge theory (the first factor at level $+k$, the second at $-k$) with two bifundamental hypermultiplets in the conjugate representations

$$A_i \in (\mathbf{N}, \overline{\mathbf{N}}), \qquad B_j \in (\overline{\mathbf{N}}, \mathbf{N}), \qquad i, j = 1, 2, \tag{I.6.47}$$

coupled through a quartic superpotential. It is $\mathcal{N} = 6$ superconformal for generic k, and its supersymmetry enhances to $\mathcal{N} = 8$ at the special levels $k = 1, 2$. The two Chern-Simons levels are opposite in sign, and each must independently satisfy the parity anomaly (I.6.43) for the bifundamental matter,

$$\big(k, -k\big), \qquad k + \tfrac{1}{2}\, T(R) \in \mathbb{Z}, \tag{I.6.48}$$

which is compatible with the paired bifundamental content in this convention. ABJM is the worldvolume theory of N M2-branes probing the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ (constructed in later volumes, Volume VIII); here it is the main Chern–Simons-matter example.

The reason it is the main example is a single number, the growth of its free energy with the rank N. Localization and its M-theory limit give the leading round-sphere free energy at fixed positive k and large N

$$F_{S^3} \;=\; \frac{\pi\sqrt{2}}{3}\, k^{1/2}\, N^{3/2} + o(N^{3/2}). \tag{I.6.49}$$

The leading exponent is $3/2$, rather than the naive 2 of a gauge theory with $O(N^2)$ matrix degrees of freedom; the $N^{3/2}$ growth is the field-theory signature of N M2-branes, the microscopic count of degrees of freedom on the M2-brane worldvolume. For sufficiently large integer N, the positive free energy obeys

$$\lim_{N\to\infty} \frac{\log F(N)}{\log N} \;=\; \frac{3}{2}, \tag{I.6.50}$$

since $\log F(N) = \frac{3}{2}\log N + \log(\pi\sqrt{2k}/3) + o(1)$ at fixed $k > 0$. This reads off the leading power without differentiating the remainder. To illustrate the arithmetic of this leading term, substitute $k = 2$ and $N = 4$ into it. This substitution is not an exact finite-rank free energy:

$$F_{\text{lead}}(2, 4) \;=\; \frac{\pi\sqrt{2}}{3}\,\sqrt{2}\cdot 4^{3/2} \;=\; \frac{\pi\sqrt{2}}{3}\,\sqrt{2}\cdot 8 \;=\; \frac{16\pi}{3}. \tag{I.6.51}$$

The ratio of leading terms at two ranks cancels the overall constant:

$$\frac{F_{\text{lead}}(k, 4)}{F_{\text{lead}}(k, 1)} \;=\; \frac{4^{3/2}}{1^{3/2}} \;=\; 8 \quad (N^{3/2}\text{ reading}), \qquad \text{versus} \qquad 4^2 \;=\; 16 \quad (\text{wrong } N^2). \tag{I.6.52}$$

Table I.6.3 shows ratios of the leading function, not exact small-rank values. The localization derivation of the ABJM matrix model and its M-theory limit (the constant $\frac{\pi\sqrt{2}}{3}$ and the $N^{3/2}$ law) is treated in Part III.

The $N^{3/2}$ scaling is the field-theory manifestation of the M-theory origin. On the standard M2-brane branch with the usual global-form choice, the moduli space of vacua and the supersymmetry both track the orbifold:

$$\mathcal{M} \;=\; \mathrm{Sym}^N\big(\mathbb{C}^4/\mathbb{Z}_k\big), \qquad \mathcal{N} \;=\; \begin{cases} 8, & k = 1, 2, \\ 6, & k \geq 3, \end{cases} \tag{I.6.53}$$

N	$F(k,N)/F(k,1) = N^{3/2}$	the $N^{3/2}$ reading	the N^2 reading (wrong)
1	1	1	1
2	$2^{3/2} = 2\sqrt{2}$	≈ 2.83	4
4	$4^{3/2} = 8$	8	16
9	$9^{3/2} = 27$	27	81

Table I.6.3: Ratios of the leading ABJM free-energy function $F_{\rm lead}(k,N)$, which cancel its overall constant and isolate the N exponent. The entries at small N evaluate that function and are not exact finite-rank partition functions.

the configuration space of N indistinguishable M2-branes on the orbifold, with the level k the order of the orbifold action. At $k=1$ the orbifold is trivial ($\mathbb{C}^4$) and the theory is the maximally supersymmetric worldvolume theory of M2-branes in flat space; at $k=2$ the $\mathbb{Z}_2$ still preserves $\mathcal{N}=8$. The $N^{3/2}$ number is the one to carry away.

Common misconception: the ABJM gauge group is $SU(4)$ with an effective level

Two errors travel together here. First, the $\mathcal{N}=6$ R-symmetry is $SU(4)$, and it acts on the matter, but it is *not* the gauge group: the ABJM gauge group is $U(N)_k \times U(N)_{-k}$, with the declared global quotient as global-form data, and the ABJM matter weights $(1,1,1,1)/k$ lie in $U(4)$, not $SU(4)$. Second, the book quotes the *bare* level k. The parity anomaly (I.6.43) is a quantization condition on the bare level (the $k + N_f/2 \in \mathbb{Z}$ form is the fundamental special case, not the general law), a constraint, not a redefinition. The global form (the diagonal $U(1)$, the $\mathbb{Z}_k$ quotient) affects the line and monopole spectrum and the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold statement; the full line-operator classification is treated with the global-form machinery.

I.6.7 The Aharony and Giveon–Kutasov dualities

Three-dimensional $\mathcal{N}=2$ gauge theories have their own Seiberg-type infrared dualities, and the monopole operators of §I.6.3 make them differ from the four-dimensional Seiberg duality of Chapter 3 in a way worth stating precisely. There are two, distinguished by whether a Chern-Simons term is present.

Aharony duality (no Chern-Simons term). A $U(N_c)$ theory with N_f flavors (and no Chern-Simons term) is infrared-dual to a $U(N_f - N_c)$ theory with N_f dual flavors, a gauge-singlet meson M, and two extra gauge-singlet *monopole operators* $V_\pm$, all coupled through a superpotential. The rank map is

$$U(N_c) \;\longleftrightarrow\; U(N_f - N_c), \tag{I.6.54}$$

and the monopole singlets $V_\pm$ are the three-dimensional feature with no four-dimensional analog: the electric theory's monopole operators are dynamical, so the magnetic dual carries them as explicit singlet fields. The map is an involution, as any duality must be:

$$U(N_c) \;\longrightarrow\; U(N_f - N_c) \;\longrightarrow\; U\big(N_f - (N_f - N_c)\big) \;=\; U(N_c). \tag{I.6.55}$$

Giveon–Kutasov duality (Chern-Simons level k). Turn on a Chern-Simons level k. A $U(N_c)_k$ theory with N_f flavors is dual to a

$$U(N_c)_k \;\longleftrightarrow\; U\big(N_f + |k| - N_c\big)_{-k} \tag{I.6.56}$$

theory with N_f flavors and a meson, but *no* monopole singlets. The Chern-Simons term lifts the monopoles by flux attachment (I.6.42): a bare monopole is no longer gauge-neutral, so it does not survive as a light singlet, and the dual has none. The dual level flips sign to $-k$, and the rank picks up the $|k|$ shift. Re-dualizing returns $U(N_c)_k$, an involution, and the $k \to 0$ limit is the Aharony map,

$$U\big(N_f + |k| - N_c\big)_{-k} \;\xrightarrow{k\to 0}\; U(N_f - N_c), \tag{I.6.57}$$

at the level of rank arithmetic. The monopole singlets require an additional infrared limit; setting $k = 0$ in the Giveon–Kutasov matter content alone does not produce the Aharony dual.

The rank arithmetic on examples: for Aharony $U(2)$ with $N_f = 5$, the dual and its re-dual are

$$U(2) \;\xrightarrow{N_f=5}\; U(N_f - N_c) = U(3) \;\xrightarrow{N_f=5}\; U(5-3) = U(2), \tag{I.6.58}$$

back to the start. For Giveon–Kutasov take $U(2)_1$ with $N_f = 3$: with the $|k|$ shift and the level flip,

$$U(2)_1 \;\xrightarrow{N_f=3}\; U(3+1-2)_{-1} = U(2)_{-1} \;\xrightarrow{N_f=3}\; U(3+1-2)_1 = U(2)_1, \tag{I.6.59}$$

again an involution (the second step uses level $|-1| = 1$). Dropping the $|k|$ shift changes the predicted rank, although that altered arithmetic can also be involutive. Table I.6.4 collects the worked ranks.

Duality	(N_c, N_f, k)	dual rank	re-dual	involutive?
Aharony	$(2, 5, 0)$	$U(3)$	$U(2)$	yes
Aharony	$(3, 7, 0)$	$U(4)$	$U(3)$	yes
Giveon–Kutasov	$(2, 3, 1)$	$U(2)_{-1}$	$U(2)_1$	yes
Giveon–Kutasov	$(1, 4, 2)$	$U(5)_{-2}$	$U(1)_2$	yes

Table I.6.4: Worked rank maps. Aharony ($k = 0$): $U(N_c) \to U(N_f - N_c)$. Giveon–Kutasov ($k \neq 0$): $U(N_c)_k \to U(N_f + |k| - N_c)_{-k}$. Every map is involutive: re-dualizing returns $U(N_c)$ (at the original level k for Giveon–Kutasov).

Common misconception: the $3d$ dualities are the reduction of $4d$ Seiberg duality

Reducing four-dimensional Seiberg duality on a circle does not give the three-dimensional dualities on the nose. The reduction-and-flow generates extra monopole superpotential terms, so the Aharony dual carries explicit monopole singlets $V_\pm$ that the naive four-dimensional map has no room for; and a Chern-Simons level changes the rank map entirely, $U(N_c)_k \to U(N_f + |k| - N_c)_{-k}$, which has no four-dimensional counterpart. The same warning applies, even more sharply, to mirror symmetry (§I.6.9), which has no four-dimensional analog at all.

Duality	Rank map	Extra content
Aharony ($k = 0$)	$U(N_c) \to U(N_f - N_c)$	meson M and monopole singlets $V_\pm$
Giveon–Kutasov ($k \neq 0$)	$U(N_c)_k \to U(N_f + \lvert k\rvert - N_c)_{-k}$	meson M; no monopole singlets (lifted by the CS term)

Table I.6.5: The two $3d$ $\mathcal{N} = 2$ Seiberg-type dualities. Both rank maps are involutive (re-dualizing returns N_c), and the Giveon–Kutasov map reduces to the Aharony map at $k = 0$. The monopole singlets are the intrinsically three-dimensional content: dynamical in Aharony, lifted by the Chern-Simons term in Giveon–Kutasov.

The Aharony dual superpotential is where the monopole singlets earn their place. The magnetic theory is $U(N_f - N_c)$ with dual quarks $q, \widetilde{q}$, the meson M (a gauge singlet with the flavor quantum numbers of the electric meson $Q\widetilde{Q}$), and the two monopole singlets $V_\pm$, coupled by

$$W \;=\; M\, q\, \widetilde{q} \;+\; V_+\, \widetilde{V}_- \;+\; V_-\, \widetilde{V}_+, \tag{I.6.60}$$

where $\widetilde{V}_\pm$ are the monopole operators of the *magnetic* gauge group. The first term is the meson coupling of four-dimensional Seiberg duality (Chapter 3); the last two tie the electric monopole singlets to the magnetic monopoles and are the intrinsically three-dimensional part: dropping them would leave extra massless singlets and spoil the match. The field-theory content of the duality is the operator map, each gauge-invariant chiral operator of the electric theory matched to one of the magnetic theory, collected in Table I.6.6.

Electric object ($U(N_c)$, N_f flavors)	Aharony magnetic object ($U(N_f - N_c)$)
meson $Q\widetilde{Q}$	gauge singlet M
monopoles $V_\pm$	gauge singlets $V_\pm$ (dynamical, coupled in W)
magnetic monopoles $\widetilde{V}_\pm$	lifted by $W \supset V_+\widetilde{V}_- + V_-\widetilde{V}_+$
FI parameter / $U(1)_J$ topological	a flavor symmetry acting on M, $V_\pm$

Table I.6.6: The Aharony operator map. The electric mesons $Q\widetilde{Q}$ become the singlet M; the electric monopole operators $V_\pm$ appear as explicit singlets in the dual (the intrinsically $3d$ content), coupled through the dual superpotential (I.6.60); and the topological $U(1)_J$ maps to a flavor symmetry of the dual. Reducing $4d$ Seiberg duality has no room for the monopole singlets, which is why the $3d$ duality is not that reduction.

Worked instance: the evidence on a small rank

The evidence for these dualities is the three-dimensional analog of Chapter 3's four checks, named on a concrete pair: $U(1)$ with N_f flavors (Aharony), dual to $U(N_f - 1)$

with the meson and monopole singlets. *Global symmetries:* both sides carry

$$SU(N_f) \times SU(N_f) \times U(1) \times U(1)_J, \tag{I.6.61}$$

with the electric $U(1)_J$ acting as a flavor symmetry on the dual mesons and monopoles. *Chiral operators:* $Q\widetilde{Q} \mapsto M$ and $V_\pm \mapsto V_\pm$ of (I.6.60), a map with no four-dimensional analog. *Parity anomaly:* the gauge and background contact terms must be compared in the two frames; equality of bare fermion indices is not a general duality rule. *The S^3 partition function:* the two round-sphere partition functions agree as functions of the real masses m_a and the FI parameter ζ,

$$\begin{aligned} Z^{S^3}_{\text{electric}}(m_a;\zeta) &= e^{i\varphi_{\text{contact}}(m_a,\zeta)} Z^{S^3}_{\text{magnetic}}(m'_a;\zeta'), \\ (m'_a,\zeta') &= \text{the duality image of } (m_a,\zeta). \end{aligned} \tag{I.6.62}$$

an identity of matrix integrals up to the scheme-dependent contact-term phase. It tests a family of observables rather than one rank. The identity and the proofs of both dualities are treated in Part III.

Part II. The $3d$ $\mathcal{N} = 4$ world

With eight supercharges the scalar fields of the vector multiplet form an $SU(2)_C$ triplet, while hypermultiplet scalars transform under $SU(2)_H$. The dual photon completes the vector-multiplet directions to a hyperkähler Coulomb branch. This puts the quantum Coulomb branch and the hyperkähler-quotient Higgs branch in the same kind of geometry. Mirror symmetry exchanges them between two theories, so a Higgs quotient can reveal a monopole-generated Coulomb space. We first count the two branches and determine an abelian Coulomb ring, then test the exchange on the basic and self-mirror pairs.

I.6.8 The $\mathcal{N} = 4$ enhancement: two branches

Doubling the supersymmetry to $3d$ $\mathcal{N} = 4$ gives eight real supercharges, the same count as $4d$ $\mathcal{N} = 2$: dimensional reduction of a four-dimensional $\mathcal{N} = 2$ theory on a circle gives a three-dimensional $\mathcal{N} = 4$ theory, the $8Q$ row of the Chapter 1 supercharge table. The enhancement changes the R-symmetry qualitatively. Where $\mathcal{N} = 2$ has a single $U(1)_R$, the $\mathcal{N} = 4$ theory has

$$SU(2)_C \times SU(2)_H \;=\; \text{Spin}(4), \tag{I.6.63}$$

two commuting $SU(2)$ factors (Chapter 1 master table): $\mathrm{so}(4) = \mathrm{su}(2) \oplus \mathrm{su}(2)$, rank $1 + 1 = 2$, dimension $3 + 3 = 6$. The two factors act separately on the two kinds of branch. This is not a single $SU(2)_R$; the two-factor structure is the whole point, and it is what mirror symmetry will exchange.

The multiplet content, read in $\mathcal{N}=2$ language. The cleanest way to see the multiplet content is to decompose each $\mathcal{N}=4$ multiplet into $\mathcal{N}=2$ pieces, exactly parallel to Chapter 5's decomposition of $4d$ $\mathcal{N}=2$ into $\mathcal{N}=1$. There are two decompositions:

$$\mathcal{N}=4 \text{ vector } = \underbrace{\mathcal{N}=2 \text{ vector}}_{A_\mu,\ \sigma,\ \text{gaugino}} + \underbrace{\mathcal{N}=2 \text{ adjoint chiral } \Phi}_{\text{2 real scalars}}, \tag{I.6.64}$$

$$\mathcal{N}=4 \text{ hyper } = \underbrace{\mathcal{N}=2 \text{ chiral } Q}_{(R)} + \underbrace{\mathcal{N}=2 \text{ chiral } \widetilde{Q}}_{(\overline{R})}, \tag{I.6.65}$$

a pair of $\mathcal{N}=2$ chirals in conjugate representations. Read off the vector-multiplet scalars from (I.6.64): the $\mathcal{N}=2$ vector supplies the one real scalar σ of (I.6.1), and the adjoint chiral Φ supplies two more real scalars, so the $\mathcal{N}=4$ vector multiplet carries

$$\underbrace{1}_{\sigma\ (\mathcal{N}=2 \text{ vector})} + \underbrace{2}_{\Phi\ (\mathcal{N}=2 \text{ adjoint chiral})} = 3 \text{ real scalars}, \tag{I.6.66}$$

a *triplet* of $SU(2)_C$. The hypermultiplet's four real scalars (two complex, one each in Q and $\widetilde{Q}$) form a doublet of doublets under $SU(2)_H$. On the abelian Coulomb branch the photon again dualizes to the compact γ, which is *not* one of the three triplet scalars: it is the separate fourth real coordinate of the local Coulomb geometry. For a rank-1 ($U(1)$) theory the count is

$$\underbrace{3}_{SU(2)_C \text{ triplet } \sigma_a} + \underbrace{1}_{\text{dual photon } \gamma} = 4 \text{ real } = 1 \text{ quaternionic}, \tag{I.6.67}$$

four real coordinates assembling one quaternionic (hyperkähler) dimension: locally a hyperkähler $\mathbb{C}^2$. For gauge rank r the generic abelian Coulomb branch has real dimension

$$\dim_{\mathbb{R}} C = \underbrace{3r}_{\text{triplet scalars}} + \underbrace{r}_{\text{dual photons}} = 4r = 4\dim_{\mathbb{H}} C, \qquad \dim_{\mathbb{H}} C = r, \tag{I.6.68}$$

the $3r$ triplet scalars plus the r dual photons, assembling r quaternionic coordinates. The factor of 4 is the hyperkähler signature: a hyperkähler manifold has real dimension a multiple of 4. Compactifying the $4d$ $\mathcal{N}=2$ Coulomb branch (complex dimension r, real $2r$) on a circle adds the r Wilson lines and the r dual photons, promoting the $2r$-real special-Kähler branch to this $4r$-real hyperkähler one; the dual photon is the piece with no four-dimensional counterpart.

The two R-symmetry factors act on the two branches separately, the structural fact mirror symmetry exploits:

$$SU(2)_C:\ C \text{ rotated},\ H \text{ fixed}; \qquad SU(2)_H:\ H \text{ rotated},\ C \text{ fixed}. \tag{I.6.69}$$

The $SU(2)_C$ factor rotates the three triplet scalars of the vector multiplet, the $SU(2)_H$ factor the hypermultiplet scalars, and the branch each acts on is identified by which factor moves it. Because the two factors are on an equal footing in the algebra,

exchanging them is a symmetry of the abstract structure, and mirror symmetry (§I.6.9) realizes that exchange by a pair of physical theories.

Both branches are hyperkähler. The Higgs branch is the hyperkähler quotient of the hypermultiplet scalars by the gauge group, and it is classically exact, precisely the object Chapter 5 built for the four-dimensional $\mathcal{N}=2$ theory; for $U(1)$ with N_f hypers its quaternionic dimension is

$$\dim_{\mathbb{H}} H \;=\; \frac{4N_f - 3 - 1}{4} \;=\; N_f - 1, \tag{I.6.70}$$

the $4N_f$ real hypermultiplet scalars, minus the 3 real moment-map (triplet D-term) constraints, minus the 1 gauge direction, all over 4. The Coulomb branch is also hyperkähler, but *quantum-corrected* and monopole-built: the dual photon supplies the extra real direction (I.6.67) that completes the local geometry to a quaternionic space, promoting the special-Kähler Coulomb branch of Chapter 5's $4d$ $\mathcal{N}=2$ world to a hyperkähler one while the Higgs branch is unchanged. For $U(1)$ with two hypers both branches are A_1 singularities, which is why $T[SU(2)]$ can be self-mirror. For larger N_f the Higgs quotient has quaternionic dimension $N_f - 1$ and is not the same A_{N_f-1} surface as the one-dimensional Coulomb branch.

Common misconception: the $\mathcal{N}=4$ Coulomb branch is special-Kähler

The four-dimensional $\mathcal{N}=2$ Coulomb branch is special-Kähler; the three-dimensional $\mathcal{N}=4$ Coulomb branch is *hyperkähler*. It is not the four-dimensional special-Kähler object reduced naively. The dual photon supplies the fourth real coordinate (I.6.67), promoting the local geometry to quaternionic, and the branch is quantum-corrected and monopole-built rather than governed by a classical prepotential. Nor is the R-symmetry a single $SU(2)_R$: it is $SU(2)_C \times SU(2)_H$, two factors acting on the two branches separately, and this two-factor structure is what mirror symmetry exchanges.

Worked instance: the $U(1)$ Coulomb-branch chiral ring

The monopole operators do more than carry topological charge; in $\mathcal{N}=4$ they generate the Coulomb-branch chiral ring. For $U(1)$ with N_f flavors the ring is generated by the two bare monopoles V_+ (charge $+1$) and V_- (charge -1) together with the vector-multiplet complex scalar Φ (the independent adjoint chiral scalar, made from the other two real vector-multiplet scalars). The product V_+V_- has vanishing topological charge, so it can be a function of Φ; its R-charge is fixed by the monopole dimensions. Using the SQED value $\Delta(\pm 1) = N_f/2$ from (I.6.21),

$$R(V_+V_-) \;=\; R(V_+) + R(V_-) \;=\; \frac{N_f}{2} + \frac{N_f}{2} \;=\; N_f, \qquad R(\Phi) \;=\; 1, \tag{I.6.71}$$

so V_+V_- has the R-charge of Φ^{N_f}. Charge matching permits this power but does not determine its coefficient or exclude other terms. In this $\mathcal{N}=4$ abelian theory the monopole product calculation supplies the chiral-ring relation

$$V_+V_- \;=\; \Phi^{N_f}. \tag{I.6.72}$$

In coordinates $(x,y,z) = (V_+, V_-, \Phi)$ this is $xy = z^{N_f}$, the equation of the A_{N_f-1} singularity, the orbifold $\mathbb{C}^2/\mathbb{Z}_{N_f}$. The orbifold order is N_f, read off the exponent of z in the ring relation, not asserted: the $\mathbb{Z}_{N_f}$ acts on $\mathbb{C}^2 = \{(u,v)\}$ and its invariants satisfy the relation identically,

$$\begin{aligned} &(u,v) \to (\omega u, \omega^{-1} v), \quad \omega = e^{2\pi i/N_f}, \qquad x = u^{N_f},\ y = v^{N_f},\ z = uv \\ &\implies \quad xy = u^{N_f} v^{N_f} = (uv)^{N_f} = z^{N_f}. \end{aligned} \tag{I.6.73}$$

These generators give the whole invariant ring. An invariant monomial $u^a v^b$ has $a-b$ divisible by N_f. Removing $(uv)^{\min(a,b)}$ leaves a power of u^{N_f} or v^{N_f}. Conversely, reducing by $xy = z^{N_f}$ leaves monomials $x^a z^c$ or $y^b z^c$ with $b > 0$, whose images in $\mathbb{C}[u,v]$ are distinct. There is therefore no further polynomial relation. This determines the complex algebraic surface, without fixing its hyperkähler metric.

Two special cases anchor the mirror computation of the next section. For $N_f = 1$ and $N_f = 2$,

$$\begin{aligned} N_f = 1: &\quad V_+V_- = \Phi \ \Rightarrow\ xy = z, \quad \text{smooth } \mathbb{C}^2, \\ N_f = 2: &\quad V_+V_- = \Phi^2 \ \Rightarrow\ xy = z^2, \quad \mathbb{C}^2/\mathbb{Z}_2 = A_1. \end{aligned} \tag{I.6.74}$$

The $N_f = 1$ ring solves for $z = xy$, a smooth $\mathbb{C}^2$ in coordinates (x,y); the $N_f = 2$ ring $xy = z^2$ is the A_1 singular point, *not* smooth, and getting the order wrong by one misidentifies the geometry. This ring is the quantum geometry that mirror symmetry will map to a classical Higgs geometry; the general machinery (abelianization, Hilbert series) is developed in Part III and later volumes.

I.6.9 Three-dimensional mirror symmetry

Three-dimensional mirror symmetry is the sharpest duality in Volume 0. It is a duality of $\mathcal{N} = 4$ theories that *exchanges the two branches*: for a theory T and its mirror $T^\vee$,

$$C(T) \ \cong\ H(T^\vee), \qquad H(T) \ \cong\ C(T^\vee), \tag{I.6.75}$$

the Coulomb branch of one theory equal to the Higgs branch of the other and vice versa (the branch frame is always carried: $C(T^\vee) \cong H(T)$ is a claim, not a renaming). Under the exchange the two R-symmetry factors swap, $SU(2)_C \leftrightarrow SU(2)_H$, and the exchange dictionary swaps the data that source the two branches:

$$\text{FI parameters} \ \longleftrightarrow\ \text{hypermultiplet masses}, \qquad U(1)_J \ \longleftrightarrow\ \text{a flavor symmetry}. \tag{I.6.76}$$

Written as a single line, with the branch frame carried and the FI parameter ζ_T and mass $m_{T^\vee}$ named on the two theories, the mirror dictionary is

$$\begin{aligned} &\text{Coulomb branch of } T \ \leftrightarrow\ \text{Higgs branch of } T^\vee, \\ &\zeta_T \leftrightarrow m_{T^\vee}, \qquad U(1)_{J,T} \leftrightarrow U(1)_{F,T^\vee}. \end{aligned} \tag{I.6.77}$$

The dictionary is an involution: applying it twice returns each item to itself. Its computational power is the asymmetry of difficulty between the branches: the Coulomb

branch is quantum and monopole-generated, the Higgs branch a classical hyperkähler quotient, so mirror symmetry turns a hard monopole computation on one side into an easy polynomial computation on the other.

Worked instance: the basic pair and the self-mirror

Two pairs make the exchange concrete, and each is worked by computing the two branches independently and only then matching them.

The basic pair: $U(1)$ with $N_f = 1$ against a free hyper. Compute the four branch dimensions. The gauge theory has a one-dimensional (quaternionic) Coulomb branch and a point Higgs branch,

$$\dim_{\mathbb{H}} C\big(U(1), N_f{=}1\big) = 1, \qquad \dim_{\mathbb{H}} H\big(U(1), N_f{=}1\big) = N_f - 1 = 0, \tag{I.6.78}$$

while the free hyper has a one-dimensional Higgs branch (its four real scalars) and no Coulomb branch. The geometries match crosswise. From the ring (I.6.72) with $N_f = 1$ the Coulomb branch is the smooth $\mathbb{C}^2$, which equals the free hyper's Higgs branch:

$$\begin{aligned} C\big(U(1),\, N_f = 1\big) \;&=\; \mathbb{C}^2 \;=\; H(\text{free hyper}), \\ H\big(U(1),\, N_f = 1\big) \;&=\; \{\cdot\} \;=\; C(\text{free hyper}), \end{aligned} \tag{I.6.79}$$

the quantum Coulomb branch of the gauge theory equal to the classical Higgs branch of the free hyper, and the trivial branches likewise exchanged.

The self-mirror: $U(1)$ with $N_f = 2$, the theory $T[SU(2)]$. Now both branches are one-dimensional. The Coulomb branch is read from the ring, the Higgs branch from the hyperkähler quotient (Chapter 5):

$$C\big(U(1), N_f{=}2\big) \overset{(\text{I.6.72})}{=} \mathbb{C}^2/\mathbb{Z}_2 = A_1, \qquad H\big(U(1), N_f{=}2\big) \;=\; \mathbb{C}^2/\mathbb{Z}_2 = A_1. \tag{I.6.80}$$

The Higgs identification uses more than its dimension. At vanishing FI parameter the complex moment map imposes $Q_1\widetilde{Q}_1 + Q_2\widetilde{Q}_2 = 0$. The gauge-invariant meson $M_{ij} = Q_i\widetilde{Q}_j$ is then a traceless rank-one 2×2 matrix. Writing its entries as $(z, x, y, -z)$, $\det M = 0$ gives $xy = -z^2$, equivalent over $\mathbb{C}$ to the A_1 equation $xy = z^2$. This independent Higgs-branch calculation supplies the geometry matched to the monopole ring. Both branches are the same A_1 singularity $\mathbb{C}^2/\mathbb{Z}_2$, so mirror symmetry maps the theory to itself:

$$C\big(T[SU(2)]\big) \;=\; H\big(T[SU(2)]\big) \;=\; \mathbb{C}^2/\mathbb{Z}_2, \qquad T[SU(2)]^\vee = T[SU(2)]. \tag{I.6.81}$$

The branch geometries here are computed, the Coulomb branch from the monopole ring and the Higgs branch from the hyperkähler quotient, and only then matched; they are not asserted equal.

Theory T	Coulomb $C(T)$	Higgs $H(T)$	Mirror $T^\vee$
$U(1)$, $N_f = 1$	$\mathbb{C}^2$	$\{\cdot\}$	free hyper
$U(1)$, $N_f = 2$ ($T[SU(2)]$)	$\mathbb{C}^2/\mathbb{Z}_2$	$\mathbb{C}^2/\mathbb{Z}_2$	itself

Table I.6.7: The basic mirror pairs. For $N_f = 1$ the gauge theory's quantum Coulomb branch $\mathbb{C}^2$ equals the free hypermultiplet's classical Higgs branch $\mathbb{C}^2$. For $N_f = 2$, $T[SU(2)]$ is self-mirror, both branches the A_1 singularity $\mathbb{C}^2/\mathbb{Z}_2$. The Coulomb branches are read off the ring $V_+V_- = \Phi^{N_f}$; the Higgs branches from the hyperkähler quotient.

The $3d$ $\mathcal{N} = 4$ deformation dictionary

The two deformation parameters affect the branch geometry differently in a $3d$ $\mathcal{N} = 4$ gauge theory. An FI parameter resolves a Higgs quotient where one exists; a real mass can deform the Coulomb geometry. Either deformation can also lift a branch:

$$\text{FI } \zeta \text{ resolves } \textbf{Higgs}, \qquad \text{real mass } m \text{ resolves } \textbf{Coulomb}.$$

Mirror symmetry exchanges the two deformations along with the two branches:

$$\zeta(T) \;\longleftrightarrow\; m(T^\vee), \qquad H(T) \;\longleftrightarrow\; C(T^\vee).$$

So the FI parameter of T resolves the Higgs branch of T, which is the Coulomb branch of $T^\vee$, resolved there by the mirror *mass*. Do *not* say the FI parameter resolves the Coulomb branch of the same theory: it resolves that theory's Higgs branch (equivalently, the mirror's Coulomb branch).

The exchange dictionary makes the match more than a coincidence of geometries, though the basic pair realizes its deformation map degenerately. Under (I.6.76) the FI parameter ζ of the gauge theory maps to a real mass of the mirror free hyper. A nonzero FI parameter lifts the gauge theory's Coulomb branch; the mirror mass lifts the free hyper's Higgs branch. Both leave an isolated vacuum:

$$\zeta_{U(1),N_f=1} \xrightarrow{\text{mirror}} m_{\text{free hyper}}, \qquad \mathcal{M}_{\zeta\neq 0}\big(U(1), N_f{=}1\big) = \mathcal{M}_{m\neq 0}(\text{free hyper}) = \{\cdot\}. \tag{I.6.82}$$

Here $\mathcal{M}$ denotes the full space of supersymmetric vacua. The isolated gauge-theory vacuum lies on the Higgs side. To see this directly, set the complex FI parameter and all masses to zero, take $g^2 > 0$, and write $a = |Q|^2$, $b = |\widetilde{Q}|^2$ for the charged hyper. The scalar potential, in a normalization where the real moment map is $a - b$, is

$$V = \frac{g^2}{2}(a - b - \zeta)^2 + 2g^2 ab + |\vec{\phi}|^2(a + b).$$

Each term is nonnegative. On the Coulomb locus $a = b = 0$, the energy is $g^2\zeta^2/2$, so no supersymmetric Coulomb vacuum survives for $\zeta \neq 0$. Vanishing of the first two terms requires $a - b = \zeta$ and $ab = 0$. For $\zeta > 0$ this gives $a = \zeta$, $b = 0$; for $\zeta < 0$ it gives $a = 0$, $b = -\zeta$. In either case the last term sets $\vec{\phi} = 0$, and quotienting the

phase of the nonzero charged field by $U(1)$ leaves one Higgs vacuum. As $\zeta \to 0$ the charged expectation value vanishes. At exactly zero FI the potential instead permits $a = b = 0$ with arbitrary $\vec{\phi}$; the photon is un-Higgsed and its dual scalar completes the reopened Coulomb branch. The limiting isolated vacuum approaches its origin, while the zero-FI theory has the whole branch. On the mirror side the massive free hyper has potential $m^2(|q|^2 + |\widetilde{q}|^2)$, whose only vacuum for $m \neq 0$ is $q = \widetilde{q} = 0$; its flat hypermultiplet space reopens at $m = 0$. The basic pair therefore illustrates the parameter exchange by lifting the matched branches, not by resolving a nontrivial singularity. In the self-mirror $T[SU(2)]$, an FI parameter resolves its Higgs A_1 branch and maps to a mass deformation of the mirror Coulomb A_1 branch. The symmetries map the same way,

$$U(1)_J(T) \;\longleftrightarrow\; U(1)_{\text{flavor}}(T^\vee), \qquad SU(2)_C(T) \;\longleftrightarrow\; SU(2)_H(T^\vee), \tag{I.6.83}$$

the topological current with no Lagrangian origin becoming an ordinary flavor current with a manifest Noether one. Every piece of data that sources one branch on one side sources the exchanged branch on the other, which is what promotes the geometric match to a duality.

For the self-mirror $T[SU(2)]$ the dictionary closes on itself: at $N_f = 2$ the topological symmetry enhances, and the exchange of the flavor $SU(2)$ (Higgs side, rotating the two hypers) with it,

$$SU(2)_{\text{flavor}} \;\longleftrightarrow\; SU(2)_J \text{ (topological, enhanced at } N_f{=}2), \tag{I.6.84}$$

is a symmetry of the theory itself, the two factors on an equal footing.

Beyond these worked pairs, the systematic construction of mirror pairs, by brane realizations (the Hanany–Witten setup) and magnetic quivers, is developed in later volumes; the proof of mirror symmetry is treated in Part III. Volume 0 delivers the statement, the exchange dictionary, and the two worked pairs.

Exit checklist

After this chapter the reader can

1. identify the $3d$ $\mathcal{N} = 2$ multiplets, name the real scalar σ in the vector multiplet, dualize the photon to the compact dual photon γ ($\star F \sim d\gamma$), and state that a rank-1 Coulomb branch is two-real-dimensional;

2. write the topological current $j_J = \star F/2\pi$, compute the basic-monopole $U(1)_J$ charge $Q_J = 1$ from the flux quantum $\int_{S^2} F = 2\pi$, and explain why the Coulomb branch is a quantum object built by monopole operators, not a classical scalar-VEV space;

3. define a monopole operator by its GNO magnetic charge m, apply the Borokhov–Kapustin–Wu formula $\Delta(m) = \frac{1}{2}\sum_i(1 - \Delta_i)\sum_{\rho_i}|\rho_i(m)| - \sum_{\alpha>0}|\alpha(m)|$, recover the SQED value $\Delta(1) = N_f/2$ as its special case at $\Delta_i = \frac{1}{2}$, and attach flux by $q_{\text{gauge}} = km$;

4. run F-maximization, extremize $\mathrm{Re}\, F = -\log|Z[\Delta]|$ built from the one-loop function $\ell(z)$ ($\ell'(z) = -\pi z \cot \pi z$), get the free-chiral value $\Delta^* = \frac{1}{2}$ with a negative Hessian (a maximum), and state the F-theorem $F_{UV} > F_{IR}$;

5. write a Chern-Simons term with pure bosonic integer level, then impose the spin-theory parity anomaly $k + \frac{1}{2}T(R) \in \mathbb{Z}$ (allowing integer or half-integer bare k, and reducing to $k + N_f/2 \in \mathbb{Z}$ for the stated fundamental matter), and recall that ABJM is $U(N)_k \times U(N)_{-k}$ with the $\mathcal{N} = 6 \to \mathcal{N} = 8$ enhancement at $k = 1, 2$ and free energy $F = \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2}$, the $N^{3/2}$ M2-brane scaling;

6. write the Aharony rank map $U(N_c) \to U(N_f - N_c)$ (with monopole singlets) and the Giveon–Kutasov map $U(N_c)_k \to U(N_f + |k| - N_c)_{-k}$ (no monopole singlets), check both are involutive, and name the S^3 partition-function identity as the strongest evidence;

7. state the $3d$ $\mathcal{N} = 4$ R-symmetry $SU(2)_C \times SU(2)_H$, count the four real Coulomb coordinates (three triplet scalars plus the dual photon) into one quaternionic dimension, explain why both branches are hyperkähler while the $4d$ $\mathcal{N} = 2$ Coulomb branch is only special-Kähler, and read off the Coulomb ring $\mathbb{C}[V_+, V_-, \Phi]/(V_+V_- - \Phi^{N_f})$, the coordinate ring of $\mathbb{C}^2/\mathbb{Z}_{N_f}$;

8. use three-dimensional mirror symmetry as the Coulomb $\leftrightarrow$ Higgs exchange, apply the dictionary (FI $\leftrightarrow$ mass, $U(1)_J \leftrightarrow$ flavor), work the basic $N_f = 1$ pair ($C = \mathbb{C}^2 = H$ of the free hyper) and the $T[SU(2)]$ self-mirror (both branches $\mathbb{C}^2/\mathbb{Z}_2$), and state which deep theorems (F-maximization, the F-theorem, mirror symmetry, the $3d$ dualities) are cited rather than proved.

Status and sources

The chapter quotes bare Chern–Simons levels, including $U(N)_k \times U(N)_{-k}$ for ABJM, uses $j_J = \star F/2\pi$ as the topological current, and specifies the GNO or coweight lattice for the chosen gauge-group form. It uses the supercharge count of Chapter 1 and takes the quiver gauge groups to have form $\prod_i U(N_i)$. The geometric interpretation uses the indicated Hanany–Witten axes. F-maximization is compared with the a-maximization example of Chapter 3.

The multiplet assignments, symmetry enhancement and mirror identification are physical inputs. The mathematical tags apply to the conditional calculations stated below.

§I.6.1 Multiplets and the dual photon. The $3d$ $\mathcal{N} = 2$ four-supercharge count as the reduction of $4d$ $\mathcal{N} = 1$, the vector multiplet's real scalar $\sigma = A_3$ (I.6.1), the dualization of the photon $\star F \sim d\gamma$ (I.6.4), and the two-real-dimensional rank-1 Coulomb branch. The multiplet assignments and Maxwell-region description are physical inputs. [MATH] applies only to the local implication $d \star F = 0 \Rightarrow \star F$ exact on a contractible patch, and to the Gaussian elimination in (I.6.6) with the stated Euclidean action and boundary conditions. The proofs are the Poincaré lemma and the displayed variation. (Aharony, Hanany, et al. (1997) the $3d$ Coulomb-branch spine). Additional calculations include the $3d$ $\mathcal{N} = 2 = 4$ real supercharge count from the ch.1 supercharge table and the $U(1)_R$ R-symmetry. Misconception

§I.6.2 The Coulomb branch and $U(1)_J$. The topological current $j_J = \star F/2\pi$ (I.6.11), conserved by the Bianchi identity, its charge the flux (I.6.12), and the basic-monopole charge $Q_J = 1$; the FI parameter as the $U(1)_J$ background and the $U(1)_J \leftrightarrow$ flavor mirror exchange previewed. The chosen flux lattice and physical BF/FI identification are inputs. [MATH] applies to $dF = 0 \Rightarrow d \star j_J = 0$ under the stated Euclidean Hodge convention, and to $\int F = 2\pi m \Rightarrow Q_J = m$, proved by substitution. The Ward-current derivation uses the stated Gaussian functional integral; its physical applicability remains conditional. (Aharony, Hanany, et al. (1997) the topological symmetry). The stated current normalization gives charge m for flux $2\pi m$.

§I.6.3 Monopole operators and their dimensions. The monopole as a GNO-flux disorder operator (I.6.16), the topological charge $Q_J = m$, the Borokhov–Kapustin–Wu dimension (I.6.18) with the gaugino term kept separate, the worked SQED $\Delta(1) = N_f/2$ (I.6.21) as a special case, and flux attachment $q_{\text{gauge}} = km$ (I.6.26). The monopole formula and the shortening and dressing conditions are physical inputs. [MATH] applies only to the displayed weight sums once the representation weights, trial R-charges and formula are supplied; their proof is finite summation. (Borokhov, Kapustin, and X.-k. Wu (2002) treats nonsupersymmetric disorder operators; Borokhov (2004) treats nonabelian monopoles). The direct SQED source is Borokhov, Kapustin, and X. Wu (2002), §2.3, p. 11 and §3.2, p. 17 (printed pages of arXiv v2): $E = |m|N_f/2$ for paired unit charges at large N_f. In $\mathcal{N} = 4$, the fixed R-symmetry and shortening underlie its continuation beyond that expansion; generic $\mathcal{N} = 2$ requires the actual infrared R-charge. Additional calculations include $\Delta(1) = N_f/2$ evaluated from the SQED matter content over $N_f = 1 \ldots 5$, the Δ_i-dependence $\Delta(1) = N_f(1 - \Delta_i)$, the gaugino term subtracting $|\alpha(m)|$, and flux attachment. The radial-quantization derivation is routed. Misconception

§I.6.4 F-maximization. The sphere free energy $F = -\log|Z[\Delta]|$ built from the one-loop function $\ell(z)$ (I.6.27) with $\ell'(z) = -\pi z \cot \pi z$ (I.6.28), the free energy (I.6.29), the free-chiral extremum $\Delta^* = \frac{1}{2}$ with Hessian $-\pi^2/2 < 0$ (I.6.31) (a maximum, the $3d$ $\Delta = R$ chiral-primary relation), the signpost to a-maximization (Chapter 3), and the F-theorem $F_{UV} > F_{IR}$. The localization formula, physical extremization principle and RG monotonicity remain stated physical inputs. [MATH] applies to the free-chiral stationary point and Hessian, obtained by differentiating the given smooth ℓ function near $\Delta = 1/2$. (Jafferis (2012) F-maximization; Kapustin, Willett, et al. (2010) the S^3 localization; Pufu (2017) the F-theorem review). Additional calculations include the ℓ' identity at rational points, the free-chiral $\Delta^* = \frac{1}{2}$ solved from the assembled free energy with a negative Hessian, and $F(\frac{1}{2})$ the local maximum. The localization derivation of $Z[\Delta]$ is routed.

§I.6.5 Chern-Simons-matter and the parity anomaly. The Chern-Simons action (I.6.39) with integer pure bosonic level, the spin-theory refinement by the parity anomaly $k + \frac{1}{2}T(R) \in \mathbb{Z}$ (I.6.43) (so the bare level may be integer or half-integer, not arbitrary fractional), the topological photon mass, and flux attachment, reducing to $k + N_f/2 \in \mathbb{Z}$ on the stated fundamental example. The parity-anomaly condition and physical level quantization are inputs. [MATH] applies only to $Q = km$ given the displayed abelian Gauss law and flux lattice, and to the integer/half-integer arithmetic given the stated representation index. (Redlich (1984) the parity anomaly). Additional calculations include pure integer normalization (with $k = 3/4$ rejected), $T(R) = N_f$, the odd-N_f half-integer / even-N_f integer admissibility computed from $T(R)$, and flux attachment.

§I.6.6 ABJM and the $N^{3/2}$ free energy. ABJM $U(N)_k \times U(N)_{-k}$ (I.6.46) with bifundamental hypers, $\mathcal{N} = 6$ superconformal, the $\mathcal{N} = 6 \to \mathcal{N} = 8$ enhancement at $k = 1, 2$, the M2-brane at $\mathbb{C}^4/\mathbb{Z}_k$, and the leading fixed-$k$, large-$N$ free energy $F \sim \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2}$ (I.6.49) with the

$N^{3/2}$ M2-brane scaling. ABJM dynamics, enhancement, M2 interpretation and the asymptotic coefficient remain stated physical inputs. [MATH] applies only to the exponent and ratios of the supplied leading function at fixed positive k, together with the displayed logarithmic-ratio limit, proved by taking the logarithm of the supplied positive asymptotic and by substitution; it says nothing about exact small-N free energies. (Aharony, Bergman, et al. (2008) ABJM; Drukker et al. (2011) the $N^{3/2}$ free energy). Additional calculations include the N-exponent 3/2 read off the imported leading expression (not 2), the levels $(k, -k)$, the formal leading-term value $F_{\text{lead}}(2, 4) = 16\pi/3$, the leading-term ratio $F_{\text{lead}}(4)/F_{\text{lead}}(1) = 8$ (not 16), and the enhancement set $\{1, 2\}$. The localization matrix model and its M-theory limit are routed.

§I.6.7 Aharony and Giveon–Kutasov dualities. The Aharony rank map $U(N_c) \to U(N_f - N_c)$ with monopole singlets (I.6.54), the Giveon–Kutasov map $U(N_c)_k \to U(N_f + |k| - N_c)_{-k}$ without them (I.6.56) (Table I.6.5), the rank-map involutions, the $k \to 0$ rank-formula limit, and the evidence (global symmetries, the chiral-operator/monopole map, contact-term matching, and the S^3 partition-function identity). The infrared transfers and operator maps have unresolved DUAL classification: no covering chain IDs are supplied here. [MATH] applies only to the rank-map involutions for integers N_c, N_f, k with nonnegative displayed ranks. Substitution proves them; it does not prove duality. (Aharony (1997) Aharony duality; Giveon and Kutasov (2009) Giveon–Kutasov duality; Willett and Yaakov (2020), §3.1 and §4, the mass/FI-deformed partition-function identities and relative phase). Additional calculations include both rank maps computed and involutive, the GK dual level $-k$, the $k = 0$ rank equality, and the hardcoded monopole-singlet presence/absence. The proofs and the partition-function-identity machinery are routed.

§I.6.8 The $\mathcal{N} = 4$ twin hyperkähler branches. The $8Q$ count as the reduction of $4d$ $\mathcal{N} = 2$, the R-symmetry $SU(2)_C \times SU(2)_H = \text{Spin}(4)$ (I.6.63), the vector multiplet's three triplet scalars with the dual photon the separate fourth Coulomb coordinate (I.6.67), both branches hyperkähler (Coulomb quantum/monopole-built, not special-Kähler), and the Coulomb relation $V_+V_- = \Phi^{N_f}$ defining the complex quotient $\mathbb{C}^2/\mathbb{Z}_{N_f}$ (I.6.72); recall the Coulomb and Higgs branches of Chapter 5. Physical multiplets, hyperkähler metrics and the monopole product are physical inputs. [MATH] applies to the dimension count on a regular free quotient stratum and to the complex invariant-ring calculation under the stated cyclic action. The displayed monomial argument proves the ring statement, not a metric identification. (Intriligator and Seiberg (1996) the $\mathcal{N} = 4$ branches; Borokhov, Kapustin, and X. Wu (2002), §3.3, pp. 20–21, equation (3), the product $V_+V_- \sim \Phi^{N_f}$ modulo descendants and its large-N_f nonzero-coefficient argument. Here SQED has N_f charge-paired hypers, zero masses, FI and CS coupling; a nonzero operator rescaling sets the coefficient to one. The algebraic ring does not determine a hyperkähler metric. The same source's §4.1, pp. 22–23 explains charge non-renormalization, while §4.2, p. 23 assumes convergence of the $1/N_f$ expansion at $N_f = 1$ for its free-mirror inference; Tachikawa (2014) the $4d$ $\mathcal{N} = 2$ reduction). Additional calculations include the $3d$ $\mathcal{N} = 4 = 8$ count and $SU(2)_C \times SU(2)_H$ (so(4), rank 2, dim 6), the three-scalar triplet with the dual photon separate, the rank-1 four-real quaternionic Coulomb count, the Coulomb branch hyperkähler-not-special-Kähler, the Higgs $\dim_{\mathbb{H}} = N_f - 1$, and the ring $xy = z^{N_f}$ for the complex quotient $\mathbb{C}^2/\mathbb{Z}_{N_f}$.

§I.6.9 Three-dimensional mirror symmetry. The Coulomb ↔ Higgs exchange (I.6.75), the swap $SU(2)_C \leftrightarrow SU(2)_H$ and the dictionary (FI ↔ mass, $U(1)_J$ ↔ flavor) (I.6.76), the basic $N_f = 1$ pair $C = \mathbb{C}^2 = H$ of the free hyper (I.6.79), and the $T[SU(2)]$ self-mirror (both branches $\mathbb{C}^2/\mathbb{Z}_2$), Table I.6.7. Mirror equivalence and the deformation dynamics are physical premises. [MATH] applies only to the complex branch rings conditional on the supplied monopole product and Higgs moment-map quotient. The $N_f = 2$ meson derivation proves the algebraic A_1

equation; mirror equivalence and its metric are separate inputs. (Intriligator and Seiberg (1996) $3d$ mirror symmetry; Boer et al. (1997) the brane realization). The two $N_f = 1$ branches are $\mathbb{C}^2$. In the $T[SU(2)]$ example the two complex branches are $\mathbb{C}^2/\mathbb{Z}_2$ with dimension $N_f - 1 = 1$. The analytic meson derivation establishes the displayed algebraic equation. The FI potential above likewise gives a conditional vacuum calculation; identifying its massive free-hyper mirror still uses the sourced duality. The agreement of complex branch equations does not establish physical mirror symmetry. The systematic construction (magnetic quivers, brane realizations) is routed to the engine volumes; the proof to Part III.

Routed (stated, not proved). F-maximization, the F-theorem, and the S^3 localization derivation of $Z[\Delta]$; the proof of three-dimensional mirror symmetry, the systematic $\mathcal{N} = 4$ Coulomb-branch geometry (abelianization, Hilbert series, magnetic quivers), the monopole-operator conformal-dimension derivation by radial quantization, and the proofs of the Aharony / Giveon–Kutasov dualities with the partition-function-identity machinery; and the gauge global-form / GNO-coweight lattice classification. These are cited physical inputs. Their fuller treatment is deferred to Part III and the engine volumes.

Further reading

Three-dimensional mirror symmetry originates in Intriligator and Seiberg (1996) and Boer et al. (1997), with the abelian case in Kapustin and Strassler (1999) and its brane realization in Hanany and Witten (1997). Monopole operators are quantized in Borokhov, Kapustin, and X.-k. Wu (2002) and their supersymmetry enhancement in Bashkirov and Kapustin (2011); the Coulomb-branch Hilbert series in Cremonesi et al. (2014) and the full branch geometry in Bullimore et al. (2017). F-maximization is treated in Jafferis (2012) and Closset, Dumitrescu, et al. (2012). The ABJM theory appears in Aharony, Bergman, et al. (2008), with the Bagger–Lambert–Gustavsson construction Bagger and Lambert (2007) and Gustavsson (2009) and the boundary theory $T[SU(N)]$ in Gaiotto and Witten (2009). Three-dimensional dualities are surveyed in Aharony, Razamat, et al. (2013), and supersymmetric partition functions and indices in Benini and Zaffaroni (2015), Closset and Kim (2016), Dimofte et al. (2014), and Gaiotto and Koroteev (2013).

For modern perspectives, see Webster and Yoo (2023) for the Higgs/Coulomb-branch formulation of three-dimensional mirror symmetry and Benvenuti et al. (2021) for monopole maps in $\mathcal{N} = 2$ quiver dualities. Magnetic-quiver technology and modern tests of $3d$ $\mathcal{N} = 4$ dualities using sphere partition functions, supersymmetric indices, and line defects are developed in Bourget et al. (2020) and Nawata et al. (2022).

References

Aharony, O. (1997). "IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c)) gauge theories". In: *Phys. Lett. B* 404, pp. 71–76. DOI: 10.1016/S0370-2693(97)00530-3. arXiv: hep-th/9703215.

Aharony, O., O. Bergman, et al. (2008). "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals". In: *JHEP* 10, p. 091. DOI: 10.1088/1126-6708/2008/10/091. arXiv: 0806.1218 [hep-th].

Aharony, O., A. Hanany, et al. (1997). "Aspects of N=2 supersymmetric gauge theories in three-dimensions". In: *Nucl. Phys. B* 499, pp. 67–99. DOI: 10.1016/S0550-3213(97)00323-4. arXiv: hep-th/9703110.

Aharony, O., S. S. Razamat, et al. (2013). "3d dualities from 4d dualities". In: *JHEP* 07, p. 149. DOI: 10.1007/JHEP07(2013)149. arXiv: 1305.3924 [hep-th].

Bagger, J. and N. Lambert (2007). "Modeling Multiple M2's". In: *Phys. Rev. D* 75, p. 045020. DOI: 10.1103/PhysRevD.75.045020. arXiv: hep-th/0611108.

Bashkirov, D. and A. Kapustin (2011). "Supersymmetry enhancement by monopole operators". In: *JHEP* 05, p. 015. DOI: 10.1007/JHEP05(2011)015. arXiv: 1007.4861 [hep-th].

Benini, F. and A. Zaffaroni (2015). "A topologically twisted index for three-dimensional supersymmetric theories". In: *JHEP* 07, p. 127. DOI: 10.1007/JHEP07(2015)127. arXiv: 1504.03698 [hep-th].

Benvenuti, S., I. Garozzo, and G. Lo Monaco (2021). "Monopoles and dualities in 3d $\mathcal{N} = 2$ quivers". In: *JHEP* 10, p. 191. DOI: 10.1007/JHEP10(2021)191. arXiv: 2012.08556 [hep-th].

Boer, J. de et al. (1997). "Mirror symmetry in three-dimensional gauge theories, quivers and D-branes". In: *Nucl. Phys. B* 493, pp. 101–147. DOI: 10.1016/S0550-3213(97)00125-9. arXiv: hep-th/9611063.

Borokhov, V. (2004). "Monopole operators in three-dimensional N=4 SYM and mirror symmetry". In: *JHEP* 03, p. 008. DOI: 10.1088/1126-6708/2004/03/008. arXiv: hep-th/0310254.

Borokhov, V., A. Kapustin, and X.-k. Wu (2002). "Topological disorder operators in three-dimensional conformal field theory". In: *JHEP* 11, p. 049. DOI: 10.1088/1126-6708/2002/11/049. arXiv: hep-th/0206054.

Borokhov, V., A. Kapustin, and X. Wu (2002). "Monopole Operators and Mirror Symmetry in Three Dimensions". In: *JHEP* 12, p. 044. DOI: 10.1088/1126-6708/2002/12/044. arXiv: hep-th/0207074.

Bourget, A. et al. (2020). "Magnetic quivers for rank 1 theories". In: *JHEP* 09, p. 189. DOI: 10.1007/JHEP09(2020)189. arXiv: 2006.16994 [hep-th].

Bullimore, M., T. Dimofte, and D. Gaiotto (2017). "The Coulomb Branch of 3d $\mathcal{N} = 4$ Theories". In: *Commun. Math. Phys.* 354.2, pp. 671–751. DOI: 10.1007/s00220-017-2903-0. arXiv: 1503.04817 [hep-th].

Closset, C., T. T. Dumitrescu, et al. (2012). "Contact Terms, Unitarity, and F-Maximization in Three-Dimensional Superconformal Theories". In: *JHEP* 10, p. 053. DOI: 10.1007/JHEP10(2012)053. arXiv: 1205.4142 [hep-th].

Closset, C. and H. Kim (2016). "Comments on twisted indices in 3d supersymmetric gauge theories". In: *JHEP* 08, p. 059. DOI: 10.1007/JHEP08(2016)059. arXiv: 1605.06531 [hep-th].

Cremonesi, S., A. Hanany, and A. Zaffaroni (2014). "Monopole operators and Hilbert series of Coulomb branches of 3d $\mathcal{N} = 4$ gauge theories". In: *JHEP* 01, p. 005. DOI: 10.1007/JHEP01(2014)005. arXiv: 1309.2657 [hep-th].

Dimofte, T., D. Gaiotto, and S. Gukov (2014). "Gauge Theories Labelled by Three-Manifolds". In: *Commun. Math. Phys.* 325, pp. 367–419. DOI: 10.1007/s00220-013-1863-2. arXiv: 1108.4389 [hep-th].

Drukker, N., M. Marino, and P. Putrov (2011). "From weak to strong coupling in ABJM theory". In: *Commun. Math. Phys.* 306, pp. 511–563. DOI: 10.1007/s00220-011-1253-6. arXiv: 1007.3837 [hep-th].

Gaiotto, D. and P. Koroteev (2013). "On Three Dimensional Quiver Gauge Theories and Integrability". In: *JHEP* 05, p. 126. DOI: 10.1007/JHEP05(2013)126. arXiv: 1304.0779 [hep-th].

Gaiotto, D. and E. Witten (2009). "S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory". In: *Adv. Theor. Math. Phys.* 13.3, pp. 721–896. DOI: 10.4310/ATMP.2009.v13.n3.a5. arXiv: 0807.3720 [hep-th].

Giveon, A. and D. Kutasov (2009). "Seiberg Duality in Chern-Simons Theory". In: *Nucl. Phys. B* 812, pp. 1–11. DOI: 10.1016/j.nuclphysb.2008.09.045. arXiv: 0808.0360 [hep-th].

Gustavsson, A. (2009). "Algebraic structures on parallel M2-branes". In: *Nucl. Phys. B* 811, pp. 66–76. DOI: 10.1016/j.nuclphysb.2008.11.014. arXiv: 0709.1260 [hep-th].

Hanany, A. and E. Witten (1997). "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics". In: *Nucl. Phys. B* 492, pp. 152–190. DOI: 10.1016/S0550-3213(97)00157-0. arXiv: hep-th/9611230.

Intriligator, K. A. and N. Seiberg (1996). "Mirror symmetry in three-dimensional gauge theories". In: *Phys. Lett. B* 387, pp. 513–519. DOI: 10.1016/0370-2693(96)01088-X. arXiv: hep-th/9607207.

Jafferis, D. L. (2012). "The Exact Superconformal R-Symmetry Extremizes Z". In: *JHEP* 05, p. 159. DOI: 10.1007/JHEP05(2012)159. arXiv: 1012.3210 [hep-th].

Kapustin, A. and M. J. Strassler (1999). "On mirror symmetry in three-dimensional Abelian gauge theories". In: *JHEP* 04, p. 021. DOI: 10.1088/1126-6708/1999/04/021. arXiv: hep-th/9902033.

Kapustin, A., B. Willett, and I. Yaakov (2010). "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter". In: *JHEP* 03, p. 089. DOI: 10.1007/JHEP03(2010)089. arXiv: 0909.4559 [hep-th].

Nawata, S. et al. (2022). "Magnetic quivers and line defects: On a duality between $3d$ $\mathcal{N} = 4$ unitary and orthosymplectic quivers". In: *JHEP* 02, p. 174. DOI: 10.1007/JHEP02(2022)174. arXiv: 2111.02831 [hep-th].

Pufu, S. S. (2017). "The F-Theorem and F-Maximization". In: *J. Phys. A* 50.44, p. 443008. DOI: 10.1088/1751-8121/aa6765. arXiv: 1608.02960 [hep-th].

Redlich, A. N. (1984). "Gauge Noninvariance and Parity Nonconservation of Three-Dimensional Fermions". In: *Phys. Rev. Lett.* 52, p. 18. DOI: 10.1103/PhysRevLett.52.18.

Tachikawa, Y. (2014). *N=2 supersymmetric dynamics for pedestrians*. Vol. 890. Lect. Notes Phys. Springer. DOI: 10.1007/978-3-319-08822-8. arXiv: 1312.2684 [hep-th].

Webster, B. and P. Yoo (2023). "3-Dimensional Mirror Symmetry". In: *Notices Amer. Math. Soc.* 70.9, pp. 1395–1406. DOI: 10.1090/noti2778. arXiv: 2308.06191 [math-ph].

Willett, B. and I. Yaakov (2020). "$\mathcal{N} = 2$ dualities and Z-extremization in three dimensions". In: *JHEP* 10, p. 136. DOI: 10.1007/JHEP10(2020)136. arXiv: 1104.0487 [hep-th].

Chapter I.7

$5d$ supersymmetric field theories

Reader contract. The eight supercharges of $4d$ $\mathcal{N} = 2$ also fit in five dimensions, but the extra dimension changes the geometry of the vacua. A five-dimensional vector multiplet has one real adjoint scalar, so its Coulomb branch is a real Weyl chamber. Its metric follows from a real, piecewise cubic prepotential. At the same time g^2 has negative mass dimension. The Yang–Mills description is therefore useful at low energy, while its ultraviolet behavior calls for more than perturbative gauge theory. For $SU(2)$ with suitable matter, the strongly coupled fixed point has an enhanced E_{N_f+1} flavor symmetry whose extra currents carry instanton number.

We first identify the multiplets and the real branch. The masses of W-bosons and hypermultiplets then determine the cubic prepotential and its metric. The $SU(2) + N_f$ calculation fixes the coefficient $8 - N_f$ and shows where a mass wall changes that metric. An instanton operator supplies a second way to read the fixed-point symmetry, illustrated by the $SO(10) \times U(1)_I \subset E_6$ current count. The final section relates this count to the conserved-current term of the superconformal index. Chapter 5 supplies the four-dimensional comparison; the existence and classification of the five-dimensional fixed points and the full instanton partition function lie beyond the calculations here.

What this chapter delivers

The $5d$ $\mathcal{N}=1$ vector and hypermultiplet, and the irrelevant coupling $[g^2]=-1$ that makes $1/g_0^2$ a mass parameter (§I.7.1); the real Coulomb branch as a Weyl chamber, contrasted with the complex special-Kähler branch of $4d$ $\mathcal{N}=2$ (§I.7.2); the real cubic prepotential $\mathcal{F}(\phi)$, classical plus one-loop exact, with the instanton data held separate, and its second-derivative metric worked on $SU(2)+N_f$ to the leading cubic coefficient $8-N_f$ (§I.7.3); instanton operators, the topological $U(1)_I$ current $j_I=\frac{1}{8\pi^2}\mathrm{Tr}(F\wedge F)$, and the E_{N_f+1} flavor enhancement counted through the $N_f=5\to E_6$ branching $\mathbf{78}=\mathbf{45}+\mathbf{1}+\mathbf{16}+\overline{\mathbf{16}}$ (§I.7.4); the $5d$ superconformal fixed points, the no-Lagrangian fact, and the boundary fact that there is no interacting $5d$ $\mathcal{N}=2$ superconformal theory (§I.7.5); and the $5d$ superconformal index on $S^4\times S^1$ in which the enhancement is visible as the adjoint character of E_{N_f+1} (§I.7.6).

I.7.1 Multiplets and the irrelevant coupling

A $5d$ $\mathcal{N}=1$ theory carries eight real supercharges. The Chapter 1 spinor table fixes the count from the minimal five-dimensional spinor and reads off the R-symmetry in one line,

$$\begin{aligned}\#\mathcal{Q} \;&=\; (\text{min } 5d \text{ spinor})\times\mathcal{N} \;=\; 8\times 1 \;=\; 8,\\ R \;&=\; SU(2)_R \;=\; \mathrm{Spin}(3),\quad \dim SU(2)_R=3.\end{aligned}\tag{I.7.1}$$

The same eight land on $4d$ $\mathcal{N}=2$ (Chapter 5), $6d$ $(1,0)$ (next chapter), and $3d$ $\mathcal{N}=4$ (Chapter 6):

$$5d\;\mathcal{N}=1 \;=\; 4d\;\mathcal{N}=2 \;=\; 6d\;(1,0) \;=\; 3d\;\mathcal{N}=4 \;=\; 8\;\mathcal{Q}.\tag{I.7.2}$$

The count identifies which four- and six-dimensional multiplets can arise by reduction.

Two multiplets carry the supersymmetry. The *vector* multiplet is a gauge field A_μ, one *real* adjoint scalar ϕ, and a symplectic-Majorana gaugino in the $SU(2)_R$ doublet,

$$V \;=\; (A_\mu,\;\phi,\;\lambda),\qquad \phi=\phi^\dagger\in\mathrm{adj}(G),\tag{I.7.3}$$

with ϕ the single scalar that parametrizes the Coulomb branch of §I.7.2. The *hypermultiplet* is four real scalars, packaged as an $SU(2)_R$ doublet of complex scalars, together with a Dirac fermion,

$$H \;=\; (q^a,\;\psi),\qquad a=1,2\ (\text{the } SU(2)_R \text{ doublet}),\tag{I.7.4}$$

identical to the $4d$ $\mathcal{N}=2$ hyper, since its scalars are R-symmetry data, not spacetime data. A fundamental hyper of G sits in the fundamental of G; one full flavor of $SU(2)$ is one hyper in the doublet.

The on-shell degrees of freedom balance in each multiplet. The vector has a gauge field with $5-2=3$ transverse polarizations plus the one real scalar, matched by the four real states of the symplectic-Majorana gaugino,

$$n_B^V \;=\; \underbrace{(5-2)}_{A_\mu}+\underbrace{1}_{\phi} \;=\; 4 \;=\; n_F^V \;=\; 4\;(\lambda).\tag{I.7.5}$$

The hyper matches its four real scalars against the four real states of the Dirac fermion,

$$n_B^H \;=\; 4\;(q^a) \;=\; n_F^H \;=\; 4\;(\psi). \tag{I.7.6}$$

The matching confirms that each multiplet closes with the stated on-shell content.

The single real scalar of (I.7.3) is the source of every difference this chapter accumulates against Chapter 5, whose $\mathcal{N}=2$ vector carries a *complex* adjoint scalar. The apparent missing scalar reappears on a circle: reducing on radius R, the gauge field acquires a fifth component A_5, a periodic scalar, and the pair assembles the $4d$ complex adjoint,

$$\Phi_{4d} \;=\; \phi + iA_5, \qquad A_5 \sim A_5 + 1/R, \tag{I.7.7}$$

so the $5d$ real Coulomb branch of §I.7.2 becomes the $4d$ complex branch on compactification, with A_5 the imaginary part. In five dimensions proper there is one real scalar, and the branch is genuinely real.

The sign flip in the coupling. The decisive structural fact of five dimensions is dimensional. The Yang–Mills density $\frac{1}{g^2}\mathrm{Tr}\,F^2$ must have mass dimension d, and $[F_{\mu\nu}]=2$ gives $[\mathrm{Tr}\,F^2]=4$, so the inverse coupling carries the rest,

$$\left[\tfrac{1}{g^2}\,\mathrm{Tr}\,F^2\right] = d \quad\Longrightarrow\quad \left[\tfrac{1}{g^2}\right] = d-4, \qquad [g^2] = 4-d. \tag{I.7.8}$$

Applying (I.7.8) in three dimensions gives

$$[g^2] \;=\; 4-d \;=\; \begin{cases} +1, & d=3 \text{ (relevant, super-renormalizable)},\\ \;\;0, & d=4 \text{ (marginal, logarithmic running)},\\ -1, & d=5 \text{ (irrelevant, non-renormalizable)}. \end{cases} \tag{I.7.9}$$

Only $d=4$ sits at the marginal edge. In five dimensions $[g^2]=-1$, so the inverse coupling

$$[1/g_0^2] \;=\; d-4 \;=\; +1 \tag{I.7.10}$$

carries mass dimension one: it is a dimensionful mass parameter, not a coupling to be tuned. The Yang–Mills theory is then non-renormalizable and serves as a low-energy effective description. For the $SU(2)+N_f$ families considered below, known superconformal fixed points provide ultraviolet completions. Dimensional analysis alone does not establish that a fixed point exists.

Because (I.7.10) makes $1/g_0^2$ a mass, it is precisely the real mass of the topological $U(1)_I$ symmetry of §I.7.4: turning it on is a relevant deformation from the fixed point to the gauge-theory regime. A separate vev of ϕ selects a vacuum on the Coulomb branch. The inverse coupling is not a Coulomb-branch modulus; the branch is parametrized by ϕ, while $1/g_0^2$ is a fixed external parameter, the coefficient of the quadratic term in the prepotential of §I.7.3. There is no $5d$ holomorphic coupling τ: the classical data is the single real number $1/g_0^2$, and the branch dependence is carried entirely by ϕ.

I.7.2 The real Coulomb branch

Give the real adjoint scalar ϕ of the vector multiplet a vacuum expectation value in the Cartan subalgebra of the gauge group G. Generically this breaks G to its maximal torus,

$$G \;\longrightarrow\; U(1)^r, \qquad r = \operatorname{rank} G, \tag{I.7.11}$$

and the low-energy theory is r free abelian vector multiplets. The moduli space of these vacua is the Coulomb branch, parametrized by the r real Cartan components ϕ^i modulo the residual Weyl action. It is a *real cone of dimension* r, a Weyl chamber,

$$\dim_{\mathbb{R}} \mathcal{M}_{\text{Coulomb}} \;=\; \operatorname{rank} G. \tag{I.7.12}$$

A charged state of weight w acquires mass $|w \cdot \phi|$ on the branch; the W-bosons carry the roots, so their mass is $|\alpha \cdot \phi|$, vanishing on a root wall where the gauge symmetry re-enhances.

For $SU(2)$ the Cartan is one-dimensional, the Weyl group is $\mathbb{Z}_2$ acting by $\phi \mapsto -\phi$, and the chamber is the half-line

$$\phi \;\geq\; 0, \qquad m_W \;=\; |\alpha \cdot \phi| \;=\; 2\phi, \tag{I.7.13}$$

one ray with a single wall at the origin. At $\phi = 0$ the W-bosons become massless and the full $SU(2)$ is restored; for $\phi > 0$ the theory is a free $U(1)$ with massive charged states. This half-line is the arena for every $SU(2)$ computation in the chapter.

$SU(3)$ shows the chamber structure in rank two. Parametrize the Cartan by (ϕ_1, ϕ_2, ϕ_3) with $\sum_i \phi_i = 0$; the Weyl group is S_3 and the fundamental chamber is the ordering

$$\phi_1 \;\geq\; \phi_2 \;\geq\; \phi_3, \qquad \phi_1 + \phi_2 + \phi_3 = 0, \tag{I.7.14}$$

a two-dimensional wedge. Its walls are the root loci

$$\alpha \cdot \phi \;=\; \phi_i - \phi_j \;=\; 0 \quad\Longrightarrow\quad \{\phi_1 = \phi_2\} \,\cup\, \{\phi_2 = \phi_3\}, \tag{I.7.15}$$

where a pair of W-bosons goes massless and an $SU(2)$ subgroup reappears. The wedge interior is the generic branch, $SU(3) \to U(1)^2$, of real dimension $\operatorname{rank} SU(3) = 2$. For any G the branch is the fundamental Weyl chamber, a real cone of dimension $\operatorname{rank} G$, walled by the root loci.

Matter refines the tiling. A hypermultiplet of weight w and real mass m_f becomes massless on the weight wall

$$w \cdot \phi + m_f \;=\; 0 \quad\xrightarrow{\;SU(2)\text{ fund }(w=\pm 1)\;}\quad \phi \;=\; |m_f|, \tag{I.7.16}$$

which does not restore gauge symmetry but does change the massive spectrum being integrated out, so the low-energy prepotential of §I.7.3 is a different cubic on either side. The branch is tiled into sub-chambers by the root and weight walls together, and $\mathcal{F}$ is piecewise cubic.

The real dimension (I.7.12) is the second structural break from Chapter 5. The $4d$ $\mathcal{N} = 2$ branch is *complex* of complex dimension r, real dimension $2r$, carrying the

special-Kähler geometry of the Seiberg–Witten curve. The $5d$ branch is real, dimension r; the missing factor of two is exactly the circle holonomy of (I.7.7),

$$\dim_{\mathbb{R}} \mathcal{M}^{5d} = r \xrightarrow{\times S^1,\ \phi\to\phi+iA_5} \dim_{\mathbb{C}} \mathcal{M}^{4d} = r, \tag{I.7.17}$$

so a $5d$ real branch of dimension r complexifies to a $4d$ complex branch of complex dimension r. In five dimensions itself the branch is real, and its low-energy data is the real cubic prepotential of §I.7.3, not a holomorphic special geometry.

Common misconception: the $5d$ Coulomb branch is the complex special-Kähler branch of $4d$ $\mathcal{N}=2$

It is tempting to carry over the $4d$ $\mathcal{N}=2$ picture wholesale: a complex Coulomb branch with a holomorphic prepotential, a Seiberg–Witten curve, and a marginal gauge coupling τ. All three fail in five dimensions. The $5d$ $\mathcal{N}=1$ Coulomb branch is *real*, parametrized by the single real adjoint scalar ϕ of each Cartan vector multiplet; it is a real cone, a Weyl chamber, not the complex special-Kähler manifold of $4d$ $\mathcal{N}=2$ (which carries the holomorphic prepotential and the Seiberg–Witten curve of Chapter 5). The $5d$ gauge coupling g^2 has mass dimension -1, so the Yang–Mills term is irrelevant and $1/g_0^2$ is a dimensionful mass parameter (the real mass of the topological $U(1)_I$ symmetry, a relevant deformation of the ultraviolet fixed point, appearing as the bare quadratic piece of the cubic prepotential), not a marginal coupling and not a Coulomb-branch modulus. The complex special geometry of four dimensions is recovered only after compactifying on a circle, where the holonomy A_5 complexifies ϕ.

I.7.3 The cubic prepotential and its metric

The low-energy effective action on the $5d$ Coulomb branch is controlled by a single real function $\mathcal{F}(\phi)$, the prepotential. It plays the same role as the holomorphic prepotential of $4d$ $\mathcal{N}=2$ (Chapter 5), encoding the branch metric as a second derivative, but it is a *real cubic* rather than an arbitrary holomorphic function.

What fixes the cubic and its one-loop coefficient? The prepotential enters the two-derivative effective Lagrangian through

$$\mathcal{L}_{\text{eff}} \supset \mathcal{F}_{ij}\,\partial\phi^i \wedge *\,\partial\phi^j + \mathcal{F}_{ijk}\,A^i \wedge F^j \wedge F^k, \qquad \mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial\phi^i \partial\phi^j \partial\phi^k}, \tag{I.7.18}$$

so $\mathcal{F}_{ijk}$ is a Chern–Simons coupling: dimensionless, and quantized for the classical piece. Since $[\phi]=1$ and $[\mathcal{F}_{ijk}]=0$, dimensional analysis caps the degree,

$$[\mathcal{F}]=3, \quad [\mathcal{F}_{ijk}]=0 \quad\Longrightarrow\quad \deg_\phi \mathcal{F} \le 3, \tag{I.7.19}$$

within the local two-derivative vector-multiplet action. A higher power of the scalar would need a coefficient of negative mass dimension; dimensional analysis alone does not exclude such a term in a general effective theory. Supersymmetry and the quantized Chern–Simons coupling supply the additional restriction.

Why the cubic is one-loop exact

The cubic describes the local two-derivative vector-multiplet action. Its one-loop exactness is a supersymmetric nonrenormalization statement, resting on three inputs: eight supercharges constrain the two-derivative vector-multiplet action to the special-real form $\mathcal{F}$; *locality* plus the quantization of the $5d$ Chern–Simons level forbid a continuous renormalization of $\mathcal{F}_{ijk}$; and integrating out a massive BPS multiplet shifts the cubic Chern–Simons coupling only at *one loop*, by the $|m|^3$ term below. Higher-loop corrections are excluded from this two-derivative vector-multiplet prepotential, and higher-derivative terms belong to separate effective couplings, not to this local cubic. The instanton data of §I.7.4 is separate protected/index data, not a local polynomial tail: it is never a third additive piece of this cubic.

The one-loop shift is fixed by the same counting. A massive multiplet of mass m contributes a dimension-three, parity-even term built from m, hence $\propto |m|^3$ (the physical mass is $|m|$), with a sign set by statistics,

$$\delta\mathcal{F}_{\text{1-loop}} \;=\; \pm\,\frac{1}{12}\,|m|^3, \qquad +\text{ vector}, \quad -\text{ hyper}. \tag{I.7.20}$$

Summing (I.7.20) over the W-bosons at the roots and the quarks at the weights gives the one-loop piece below.

The prepotential has two pieces,

$$\mathcal{F}(\phi) \;=\; \underbrace{\frac{1}{2g_0^2}\,h_{ij}\,\phi^i\phi^j \;+\; \frac{\kappa}{6}\,d_{ijk}\,\phi^i\phi^j\phi^k}_{\text{classical}} \;+\; \mathcal{F}_{\text{1-loop}}(\phi). \tag{I.7.21}$$

The classical piece is the bare quadratic $\frac{1}{2g_0^2}\phi^2$, with the dimensionful inverse coupling of §I.7.1, plus a Chern–Simons cubic of level κ and symbol

$$d_{ijk} \;=\; \tfrac{1}{2}\,\mathrm{Tr}\big(T_i\{T_j,T_k\}\big), \tag{I.7.22}$$

present only for groups with a nonzero cubic Casimir, such as $SU(N\geq 3)$. Since $SU(2)$ has no cubic Casimir,

$$d_{ijk}^{SU(2)} = 0 \quad\Longrightarrow\quad \kappa_{SU(2)} = 0, \tag{I.7.23}$$

so $SU(2)$ carries no classical cubic and its entire cubic comes from the one-loop piece: this is why it is the cleanest worked example.

The one-loop piece sums (I.7.20) over the roots α of G (vector, $+$) and the weights w of each hyper R_f of real mass m_f (hyper, $-$),

$$\mathcal{F}_{\text{1-loop}}(\phi) \;=\; \frac{1}{12}\left(\sum_{\alpha\,\in\,\text{roots}} |\alpha\cdot\phi|^3 \;-\; \sum_{f}\sum_{w\,\in\,R_f} |w\cdot\phi + m_f|^3\right). \tag{I.7.24}$$

The absolute values are essential: each $|m|^3$ is a genuine cubic only inside a region where its argument keeps a fixed sign, so $\mathcal{F}$ is *piecewise cubic*, one polynomial per sub-chamber, switching across the walls where an argument vanishes,

$$\underbrace{\alpha\cdot\phi = 0}_{\text{W-boson (root) wall}}, \qquad \underbrace{w\cdot\phi + m_f = 0}_{\text{hyper (weight) wall, } \phi=|m_f|}. \tag{I.7.25}$$

Crossing such a wall is a flop of the effective theory, a genuine change of the low-energy prepotential, not a coordinate artifact.

The physical content is the second derivative, the effective gauge-coupling matrix, equivalently the branch metric,

$$\tau_{ij}(\phi) \;=\; \frac{\partial^2 \mathcal{F}}{\partial\phi^i\,\partial\phi^j}. \tag{I.7.26}$$

Unitarity requires $\tau_{ij} \succ 0$ in the physical chamber. This positivity is a real constraint, not a formality: it bounds how much matter a $5d$ gauge theory can carry, as the $SU(2)+N_f$ example now shows.

Worked instance: $SU(2)$ with N_f flavors

Take $SU(2)$ on its half-line $\phi \geq 0$ with N_f fundamental hypermultiplets. Fix the normalization once: the roots and fundamental weights are

$$\alpha\cdot\phi \;=\; \pm 2\phi \;\;(m_W = 2\phi), \qquad w \;=\; \pm 1 \;\;(m_{\text{quark}} = |\phi \pm m_f|). \tag{I.7.27}$$

The roots ± 2 and weights ± 1 specify the Cartan coordinate and charge convention used below. A consistent rescaling also rescales the bare term and leaves the physics unchanged. With $\kappa_{SU(2)} = 0$ from (I.7.23), the full prepotential is the bare quadratic plus the one-loop assembly (I.7.24),

$$\mathcal{F}(\phi) \;=\; \frac{1}{2g_0^2}\,\phi^2 \;+\; \frac{1}{12}\left(|2\phi|^3 + |-2\phi|^3 \;-\; \sum_{f=1}^{N_f} \left(|\phi + m_f|^3 + |-\phi + m_f|^3 \right) \right). \tag{I.7.28}$$

Now read the leading cubic coefficient in the large-ϕ chamber, where all the masses are negligible against ϕ and every absolute value resolves to its argument's magnitude on $\phi > 0$. The vector piece gives

$$\frac{1}{12}\left(|2\phi|^3 + |-2\phi|^3 \right) \;=\; \frac{1}{12}\,(8+8)\,\phi^3 \;=\; \frac{16}{12}\,\phi^3 \;=\; \frac{1}{6}\cdot 8\,\phi^3, \tag{I.7.29}$$

the number 8 being the sum $|2|^3 + |-2|^3 = 16$ divided by two, the 8 that starts the count. Each fundamental hypermultiplet, with its two weights ± 1, subtracts

$$\frac{1}{12}\left(|\phi|^3 + |-\phi|^3 \right) \;=\; \frac{2}{12}\,\phi^3 \;=\; \frac{1}{6}\,\phi^3, \tag{I.7.30}$$

one unit of the same $\frac{1}{6}$. Adding the vector piece and subtracting N_f copies of the hyper piece, the large-ϕ prepotential is

$$\mathcal{F}(\phi) \;=\; \frac{1}{2g_0^2}\,\phi^2 \;+\; \frac{1}{6}\,(8 - N_f)\,\phi^3 \;+\; \cdots, \tag{I.7.31}$$

so the cubic coefficient of $6\mathcal{F}$ is exactly $8 - N_f$. This follows by summing the root and weight contributions in (I.7.28); taking three derivatives gives the same coefficient. Substituting $\alpha = \pm 1$, $w = \pm\frac{1}{2}$ while holding ϕ and the bare term fixed would give

$$\tfrac{1}{12}(1+1) = \tfrac{1}{6}, \qquad \tfrac{1}{12}\cdot 2\cdot\tfrac{1}{8} = \tfrac{1}{48} \;\;\text{(per flavor)}, \tag{I.7.32}$$

a different numerical slope in that inconsistent substitution. With $\bar{\phi} = 2\phi$, the same masses have roots $\pm\bar{\phi}$ and weights $\pm\bar{\phi}/2$, while the bare term becomes $\bar{\phi}^2/(8g_0^2)$ and the cubic becomes $(8 - N_f)\bar{\phi}^3/48$. The third derivative changes by a coordinate factor of $1/8$; metric positivity and the E_{N_f+1} window do not. The integer-charge basis above is convenient for Chern–Simons quantization.

The Coulomb-branch metric is the second derivative (I.7.26) of (I.7.31),

$$\tau(\phi) \;=\; \mathcal{F}''(\phi) \;=\; \frac{1}{g_0^2} \;+\; (8 - N_f)\,\phi \;+\; \cdots, \tag{I.7.33}$$

affine in ϕ: constant term $1/g_0^2$, slope $8 - N_f$. It is the second derivative, not the first,

$$\mathcal{F}'(\phi) \;=\; \frac{\phi}{g_0^2} + \tfrac{1}{2}(8 - N_f)\phi^2 \;\neq\; \tau(\phi), \tag{I.7.34}$$

so using $\mathcal{F}'$ for the metric is an off-by-a-derivative error. Positivity on $\phi \geq 0$, with $1/g_0^2 > 0$ and $\phi \to \infty$, needs a non-negative slope,

$$\tau(\phi) > 0 \;\; \forall\, \phi \geq 0 \quad\Longleftrightarrow\quad 8 - N_f \;\geq\; 0 \quad\Longleftrightarrow\quad N_f \;\leq\; 8. \tag{I.7.35}$$

The sourced E_{N_f+1} fixed-point family lies in the stricter window $8 - N_f > 0$,

$$8 - N_f > 0 \quad\Longleftrightarrow\quad N_f \leq 7, \qquad E_{N_f+1}\big|_{N_f=7} = E_8 \text{ (the last enhanced row).} \tag{I.7.36}$$

So E_8 is the $N_f = 7$ row, where $8 - N_f = 1$. At $N_f = 8$ the coefficient hits zero: the marginal boundary, an affine-E_8 / 6d-lift situation, *not* the ordinary 5d E_8 fixed point. For $N_f \geq 9$ the coefficient is negative and the metric goes negative at large ϕ, obstructing this gauge-phase route to an ordinary fixed point. Table I.7.1 collects the chamber data.

Normalization check. A common slip double-counts the two weights of each flavor into a coefficient $8 - 2N_f$. But the two weights already sit inside one unit of $\frac{1}{6}$,

$$\tfrac{1}{12}\big(|\phi|^3 + |-\phi|^3\big) \;=\; \tfrac{1}{12}\cdot 2\,\phi^3 \;=\; \tfrac{1}{6}\,\phi^3 \quad \text{(one unit, not two)}, \tag{I.7.37}$$

so the coefficient is $8 - N_f$, not $8 - 2N_f$. The doubled version would go negative already at $N_f = 5$,

$$8 - 2N_f\big|_{N_f=5} \;=\; -2 \;<\; 0, \tag{I.7.38}$$

forbidding the E_6, E_7, E_8 theories that manifestly exist; the correct $8 - N_f$ stays non-negative through $N_f = 7$.

Worked instance: the classical cubic of $SU(3)$

The group $SU(2)$ had no classical cubic, so its entire cubic prepotential came from the one-loop piece. To see the classical Chern–Simons cubic that only larger groups carry, take pure $SU(3)$, whose rank-two Coulomb branch was the wedge (I.7.14). Parametrize

N_f	$6\mathcal{F}$ cubic coeff. $8-N_f$	metric slope $d\tau/d\phi$	metric on $\phi \geq 0$
0	8	8	positive
1	7	7	positive
2	6	6	positive
3	5	5	positive
4	4	4	positive
5	3	3	positive
6	2	2	positive
7	1	1	positive (last enhanced row, $\to E_8$)
8	0	0	flat (boundary)
9	-1	-1	negative at large ϕ (no fixed point)

Table I.7.1: The $SU(2)+N_f$ Coulomb-branch chamber data. The leading cubic coefficient of $6\mathcal{F}$ is $8-N_f$, computed by differentiating the $|m|^3$ assembly (I.7.28); it equals the slope of the affine metric $\tau(\phi) = 1/g_0^2 + (8-N_f)\phi$. Positivity on the half-line requires $8-N_f \geq 0$, i.e. $N_f \leq 8$; the strict E_{N_f+1} fixed-point window is the subwindow $8-N_f > 0$, $N_f \leq 7$, with $N_f = 7$ giving E_8, while $N_f = 8$ ($8-N_f = 0$) is the marginal boundary (affine-E_8 / $6d$-lift), not an ordinary E_9. The sub-chamber walls, where a hypermultiplet becomes massless, sit at $\phi = |m_f|$; the origin $\phi = 0$ is the W-boson wall. Every number is differentiated from the assembly, not quoted.

the Cartan by (ϕ_1, ϕ_2, ϕ_3) with $\sum_i \phi_i = 0$. The $5d$ Chern–Simons term at level κ contributes the classical cubic

$$\mathcal{F}_{\rm CS}(\phi) \;=\; \frac{\kappa}{6}\sum_{i=1}^{3}\phi_i^{\,3} \;=\; \frac{\kappa}{2}\,\phi_1\phi_2\phi_3, \tag{I.7.39}$$

where the second form uses the trace identity valid on the $SU(3)$ Cartan,

$$\phi_1+\phi_2+\phi_3 = 0 \quad\Longrightarrow\quad \phi_1^3+\phi_2^3+\phi_3^3 = 3\,\phi_1\phi_2\phi_3. \tag{I.7.40}$$

This is a genuinely cubic classical contribution, present because $SU(3)$ has a nonzero cubic Casimir; κ is a quantized level (an integer, up to a matter-set half-integer shift), not a modulus. The one-loop vector piece sums over the six roots $\pm(\phi_i-\phi_j)$,

$$\mathcal{F}^{\rm vec}_{\text{1-loop}} \;=\; \frac{1}{12}\sum_{i<j}\Big(|\phi_i-\phi_j|^3+|\phi_j-\phi_i|^3\Big) \;=\; \frac{1}{6}\sum_{i<j}|\phi_i-\phi_j|^3, \tag{I.7.41}$$

smooth inside the chamber (I.7.14) where each ordered difference keeps a fixed sign, and jumping across the walls (I.7.15). So $SU(N\geq 3)$ carries *both* the classical cubic

(I.7.39) and the one-loop cubic, whereas $SU(2)$ carries only the one-loop cubic. The metric is again the second derivative, now a rank-two matrix,

$$\tau_{ij}(\phi) \;=\; \frac{\partial^2 \mathcal{F}}{\partial\phi^i\,\partial\phi^j} \;\in\; \mathrm{Sym}_2(\mathbb{R}), \qquad \tau_{ij} \succ 0 \text{ in the chamber}, \tag{I.7.42}$$

and its positive-definiteness is the same convexity constraint that bounds the matter and the level.

Worked instance: the flop across a mass wall

The leading cubic was read at large ϕ. Keeping a mass on shows the piecewise structure explicitly. Take $SU(2)$ with one fundamental of real mass $m > 0$, quark masses $|\phi + m|$ and $|\phi - m|$, so

$$\mathcal{F}(\phi) \;=\; \frac{1}{2g_0^2}\,\phi^2 \;+\; \frac{1}{12}\Big(2|2\phi|^3 - |\phi+m|^3 - |\phi-m|^3\Big), \tag{I.7.43}$$

where $|2\phi|^3 = |-2\phi|^3$ has been combined. On the branch $\phi > 0$ the term $|\phi + m|^3 = (\phi+m)^3$ is smooth, but $|\phi - m|^3$ switches its resolution at the wall $\phi = m$: it is $(m-\phi)^3$ in the inner sub-chamber $0 < \phi < m$ and $(\phi - m)^3$ in the outer sub-chamber $\phi > m$. Resolve the two pieces. In the outer chamber $\phi > m$, with $|2\phi|^3 = 8\phi^3$,

$$\mathcal{F}_{\phi>m}(\phi) \;=\; \frac{\phi^2}{2g_0^2} \;+\; \frac{1}{12}\Big(16\phi^3 - (\phi+m)^3 - (\phi-m)^3\Big) \;=\; \frac{\phi^2}{2g_0^2} \;+\; \frac{1}{12}\big(14\phi^3 - 6m^2\phi\big), \tag{I.7.44}$$

using $(\phi+m)^3 + (\phi-m)^3 = 2\phi^3 + 6m^2\phi$. Its second derivative is the outer metric

$$\tau_{\phi>m}(\phi) \;=\; \frac{1}{g_0^2} \;+\; \frac{1}{12}\big(84\phi\big) \;=\; \frac{1}{g_0^2} \;+\; 7\phi, \tag{I.7.45}$$

whose slope is $7 = 8 - N_f$ at $N_f = 1$, matching (I.7.33). In the inner chamber $0 < \phi < m$, where $|\phi - m|^3 = (m-\phi)^3$,

$$\mathcal{F}_{\phi<m}(\phi) \;=\; \frac{\phi^2}{2g_0^2} + \frac{1}{12}\Big(16\phi^3 - (\phi+m)^3 - (m-\phi)^3\Big) \;=\; \frac{\phi^2}{2g_0^2} + \frac{1}{12}\big(16\phi^3 - 6m\phi^2 - 2m^3\big), \tag{I.7.46}$$

using $(\phi+m)^3 + (m-\phi)^3 = 6m\phi^2 + 2m^3$, and its second derivative is the inner metric

$$\tau_{\phi<m}(\phi) \;=\; \frac{1}{g_0^2} \;+\; \frac{1}{12}\big(96\phi - 12m\big) \;=\; \frac{1}{g_0^2} \;+\; 8\phi \;-\; m. \tag{I.7.47}$$

The inner slope is 8, the pure-$SU(2)$ value. The two hyper weight contributions cancel each other's cubic dependence for $0 < \phi < m$; the outer slope is $7 = 8 - N_f$ once the hyper is light. The metric is continuous at the wall while the slope jumps,

$$\tau_{\phi<m}(m) = \tfrac{1}{g_0^2} + 8m - m = \tfrac{1}{g_0^2} + 7m \;=\; \tau_{\phi>m}(m), \qquad \tau' \colon\; 8 \to 7. \tag{I.7.48}$$

This kink, continuous value and jumping slope, is the flop: a genuine change of the low-energy prepotential where the hyper becomes massless, a change in the five-dimensional effective coupling caused by a massless hypermultiplet.

I.7.4 Instanton operators and the E_n flavor enhancement

A $5d$ gauge theory carries a topological symmetry with no local $4d$ counterpart. Its current is the instanton-number four-form,

$$j_I \;=\; \frac{1}{8\pi^2}\,\mathrm{Tr}(F\wedge F), \tag{I.7.49}$$

conserved for a topological reason. In a gauge patch it is locally a Chern–Simons derivative; the Bianchi identity makes it closed globally,

$$\mathrm{Tr}(F\wedge F) \underset{\text{locally}}{=} d\,\omega_{CS} \quad\Longrightarrow\quad d\,\mathrm{Tr}(F\wedge F)=0, \quad d*(*j_I)=0, \tag{I.7.50}$$

so its Hodge dual $J_I = *j_I$ is a conserved one-form current, with $d*J_I = 0$, generating a $U(1)_I$. Closure of the four-form j_I is the conservation statement; it does not require the one-form J_I itself to be closed. The conserved charge is the integer instanton number

$$k \;=\; \frac{1}{8\pi^2}\int \mathrm{Tr}(F\wedge F) \;\in\; \mathbb{Z}. \tag{I.7.51}$$

This $U(1)_I$ is a global flavor symmetry, not an R-symmetry: the R-symmetry stays $SU(2)_R$.

The operators charged under $U(1)_I$ are the *instanton operators*, disorder operators defined by a boundary condition rather than a polynomial in the fields. An operator of charge k is inserted by demanding instanton flux k through a small surrounding four-sphere,

$$\frac{1}{8\pi^2}\int_{S^4}\mathrm{Tr}(F\wedge F) \;=\; k \quad (S^4 \text{ around the insertion}), \qquad k=1:\ \text{charge 1}. \tag{I.7.52}$$

This is the $5d$ analog of the $3d$ monopole operator of Chapter 6. The two sit in one dimensional pattern: a disorder operator in d dimensions carries the flux of a characteristic class on the surrounding S^{d-1},

$$d=3:\ \frac{1}{2\pi}\int_{S^2} F \ (\text{monopole}, U(1)_J), \qquad d=5:\ \frac{1}{8\pi^2}\int_{S^4}\mathrm{Tr}(F\wedge F)\ (\text{instanton}, U(1)_I). \tag{I.7.53}$$

What is special to five dimensions is that the flux is quadratic in F, so the surrounding sphere is four-dimensional and the charge is instanton number rather than magnetic charge. These operators act on the Hilbert space even though they are invisible in the classical Lagrangian.

There is also an instanton *particle*, a finite-mass BPS state of unit $U(1)_I$ charge. Its mass is its charge times the $U(1)_I$ real mass $1/g_0^2$, up to $8\pi^2$,

$$M_{\text{inst}} \;\sim\; \frac{8\pi^2}{g_0^2}, \qquad [M_{\text{inst}}] \;=\; [1/g_0^2] \;=\; +1, \tag{I.7.54}$$

a genuine mass in five dimensions. In a given chamber the full BPS central charge may pick up extra ϕ-dependence, but the dimensional anchor is $8\pi^2/g_0^2$. The naive form ϕ/g^2 fails the dimension test,

$$[\phi/g^2] \;=\; [\phi] + [1/g^2] \;=\; 1+1 \;=\; 2 \;\neq\; 1, \tag{I.7.55}$$

so it cannot be a $5d$ mass; the correct anchor (I.7.54) has dimension one.

The enhancement. At weak coupling the flavor symmetry rotating the hypers, times $U(1)_I$, is the perturbative group. For $SU(2) + N_f$ the flavor factor is orthogonal, not unitary, because the $SU(2)$ fundamental is pseudoreal: each full flavor splits into two half-hypers, and the symmetry rotating them while preserving the symplectic pairing is

$$N_f \text{ flavors} \;=\; 2N_f \text{ half-hypers} \quad\Longrightarrow\quad \text{flavor} = SO(2N_f) \text{ (not } U(N_f)). \tag{I.7.56}$$

The full perturbative symmetry is then

$$SO(2N_f) \;\times\; U(1)_I. \tag{I.7.57}$$

At the fixed point $1/g_0^2 \to 0$ the $k = \pm 1$ instanton currents combine with the perturbative currents to close an exceptional algebra,

$$SO(2N_f) \;\times\; U(1)_I \quad\xrightarrow{1/g_0^2 \to 0}\quad E_{N_f+1} \qquad (N_f \leq 7), \tag{I.7.58}$$

the Seiberg E_n fixed points. This is a flavor enhancement; the R-symmetry $SU(2)_R$ does not participate.

What enhances is the conserved-current content. At weak coupling the currents are those of $SO(2N_f) \times U(1)_I$, conserved for all $1/g_0^2$. At the fixed point the $k = \pm 1$ instanton operators, ordinary charged operators of nonzero dimension at generic coupling, hit the protected dimension of a $5d$ conserved-current multiplet,

$$\text{instanton currents} \;\in\; \mathbf{S}_{+1} \oplus \overline{\mathbf{S}}_{-1} \quad (\text{spinor of } SO(2N_f),\; U(1)_I \text{ charge } \pm 1), \tag{I.7.59}$$

and their currents join the perturbative ones. The enlarged set closes on E_{N_f+1} because the spinor at $U(1)_I$ charge ± 1 is exactly what completes the $SO(2N_f) \times U(1)$ adjoint to the exceptional adjoint. The enhancement recruits new currents from the instanton sector; it is not a rearrangement of the perturbative ones, and it is visible only at the fixed point.

Common misconception: the E_n enhancement enlarges the R-symmetry

The enhanced exceptional symmetry E_{N_f+1} at the $5d$ ultraviolet fixed point is a *flavor* symmetry, not an R-symmetry. The $5d$ $\mathcal{N} = 1$ R-symmetry is $SU(2)_R$ everywhere along the flow and at the fixed point; the Chapter 1 master table is explicit on this. The enhancement assembles the perturbative flavor symmetry $SO(2N_f) \times U(1)_I$, in which the $U(1)_I$ is the topological instanton-number symmetry (also a flavor symmetry, not R), with the conserved currents carried by the $k = \pm 1$ instanton operators, into the flavor group E_{N_f+1}. The R-symmetry $SU(2)_R$, of dimension three, is untouched: $\dim E_6 = 78$ is a flavor symmetry, not a three-dimensional R-symmetry. Conflating the flavor enhancement with an enlargement of the R-symmetry is the standard five-dimensional trap the master table warns against.

Worked instance: the E_6 enhancement of $SU(2) + 5$ flavors

The cleanest enhancement to count is $N_f = 5$, where $SO(10) \times U(1)_I \to E_6$. Assemble the dimensions. The perturbative flavor group $SO(10)$ has dimension

$$\dim SO(2N_f) \;=\; N_f(2N_f - 1), \qquad \dim SO(10) \;=\; 5 \cdot 9 \;=\; 45, \tag{I.7.60}$$

the antisymmetric-matrix count. Add the one $U(1)_I$ current. The instanton operators at $k = \pm 1$ supply the two chiral spinors of $SO(10)$, each of dimension

$$\dim(\text{chiral spinor of } SO(2N_f)) \;=\; 2^{\,N_f - 1}, \qquad \dim \mathbf{16} \;=\; 2^4 \;=\; 16, \tag{I.7.61}$$

so the $k = +1$ operator gives the $\mathbf{16}$ and the $k = -1$ operator the conjugate $\overline{\mathbf{16}}$. The total conserved-current count is

$$\dim SO(10) + \dim U(1)_I + \dim \mathbf{16} + \dim \overline{\mathbf{16}} \;=\; 45 + 1 + 16 + 16 \;=\; 78 \;=\; \dim E_6. \tag{I.7.62}$$

The dimension of E_6 follows from its 72 roots and rank 6. Generating the roots from its Cartan matrix gives $72 + 6 = 78$, matching the assembled left-hand side exactly. The identity has a clean group-theoretic reading: the adjoint of E_6 branches under its maximal subgroup $SO(10) \times U(1)$ as

$$\mathbf{78} \;=\; \mathbf{45}_0 \;+\; \mathbf{1}_0 \;+\; \mathbf{16}_{+1} \;+\; \overline{\mathbf{16}}_{-1}, \tag{I.7.63}$$

the perturbative adjoint $\mathbf{45}_0$ plus the $U(1)$ current $\mathbf{1}_0$ plus the two instanton spinors at $U(1)_I$ charge ± 1. The ± 1 subscripts are the instanton numbers: the $\mathbf{16}$ is carried by the $k = +1$ instanton operator and the $\overline{\mathbf{16}}$ by $k = -1$. The count connects perturbative and instanton currents. Here $\dim SO(10)$ follows from the antisymmetric tensor, $\dim \mathbf{16}$ from the spinor rank, and $\dim E_6$ from its root system.

Two structural guards. Dropping the $U(1)_I$ current, or replacing the spinor by the vector, both break the count,

$$45 + \qquad 16 + 16 = 77 \neq 78, \qquad 45 + 1 + 2 \cdot (2N_f)\big|_{N_f = 5} = 45 + 1 + 20 = 66 \neq 78, \tag{I.7.64}$$

so the topological current is essential and the instanton currents must be in the *spinor*, not the vector.

The same count closes at the low end of the window, where the exceptionals are small and recognizable,

$$\begin{aligned} N_f = 1: &\quad \dim SO(2) + 1 + 2 \cdot 2^0 = 1 + 1 + 2 = 4 = \dim \big(SU(2) \times U(1)\big) = \dim E_2, \\ N_f = 4: &\quad \dim SO(8) + 1 + 2 \cdot 2^3 = 28 + 1 + 16 = 45 = \dim SO(10) = \dim E_5, \end{aligned} \tag{I.7.65}$$

using $\dim SO(2N_f) = N_f(2N_f - 1)$ and the chiral spinor $2^{N_f - 1}$. The window is $E_{N_f + 1}$ for $N_f = 1, \ldots, 7$: the classical $E_2 = SU(2) \times U(1)$, $E_3 = SU(3) \times SU(2)$, $E_4 = SU(5)$, $E_5 = SO(10)$, then the genuine exceptionals E_6, E_7, E_8.

A caution belongs with the higher rows: only $N_f \leq 5$ closes with the simple two-spinor count. At $N_f = 6, 7$ the two basic $k = \pm 1$ spinors under-count,

$$\begin{aligned} N_f = 6: &\quad 66 + 1 + 2 \cdot 32 = 131 \;<\; 133 = \dim E_7, \\ N_f = 7: &\quad 91 + 1 + 2 \cdot 64 = 220 \;<\; 248 = \dim E_8, \end{aligned} \tag{I.7.66}$$

so the true instanton content is richer. The total nonperturbative current contribution beyond $SO(2N_f) \times U(1)_I$ is

$$\dim E_{N_f+1} - \dim SO(2N_f) - 1 \;=\; \begin{cases} 66, & N_f = 6, \\ 156, & N_f = 7, \end{cases} \tag{I.7.67}$$

of which the two basic spinors account for 64 and 128, respectively. The remaining shortfalls are 2 for E_7 and 28 for E_8, supplied by higher/dressed instanton sectors (higher instanton charge, operators dressed by the gauge and flavor data), whose full enumeration is developed in Part III and the five-dimensional construction in Volume X. The basic $k = \pm 1$ spinors alone do *not* prove E_7 or E_8; $N_f = 5 \to E_6$ is the unambiguous worked instance, and the higher rows carry the honest note that the naive count under-counts.

N_f	perturbative	enhanced E_{N_f+1}	$\dim E_{N_f+1}$	nonperturbative currents
0	$U(1)_I$	$E_1/\widetilde{E}_1$ (see below)	(n/a)	discrete-θ pair
1	$SO(2) \times U(1)$	$E_2 = SU(2) \times U(1)$	4	$2^0 + 2^0 = 2$
2	$SO(4) \times U(1)$	$E_3 = SU(3) \times SU(2)$	11	$2^1 + 2^1 = 4$
3	$SO(6) \times U(1)$	$E_4 = SU(5)$	24	$2^2 + 2^2 = 8$
4	$SO(8) \times U(1)$	$E_5 = SO(10)$	45	$2^3 + 2^3 = 16$
5	$SO(10) \times U(1)$	E_6	78	$2^4 + 2^4 = 32$ (closes)
6	$SO(12) \times U(1)$	E_7	133	66 total $(64 + 2)$
7	$SO(14) \times U(1)$	E_8	248	156 total $(128 + 28)$

Table I.7.2: The $SU(2)+N_f$ ultraviolet flavor-symmetry enhancement. The perturbative $SO(2N_f) \times U(1)_I$ enhances to E_{N_f+1} at the fixed point; the "nonperturbative currents" column is $\dim E_{N_f+1} - \dim SO(2N_f) - 1$. For $N_f \leq 5$ these are exactly the two $k = \pm 1$ chiral spinors of $SO(2N_f)$, of dimension 2^{N_f-1} each; $N_f = 5 \to E_6$ is the worked case $(45 + 1 + 16 + 16 = 78)$. At $N_f = 6, 7$ the simple two-spinor count *under-counts* $(131 < 133,\ 220 < 248)$; the total nonperturbative dimensions are 66 and 156, with extra shortfalls 2 and 28 beyond the two basic spinors supplied by higher/dressed instanton sectors. The full enumeration is developed in Part III, so the basic $k = \pm 1$ spinors alone do not prove E_7/E_8. The $N_f = 0$ row is not a single enhancement row; it is the discrete-θ pair $E_1/\widetilde{E}_1$ discussed in §I.7.5. The exceptional dimensions E_6, E_7, E_8 follow from their root systems; the lower rows use classical-group dimension formulas.

I.7.5 The $5d$ superconformal fixed points

The positive metric slope is compatible with the known ultraviolet fixed points in the $SU(2) + N_f$ sequence. In those theories, $1/g_0^2 \to 0$ reaches a $5d$ *superconformal field*

theory; the prepotential test alone cannot establish its existence. The gauge theory is not its own completion; it is a relevant (mass) deformation of the fixed point by the $U(1)_I$ mass $1/g_0^2$,

$$\text{SCFT } (E_{N_f+1}) \xrightarrow[\text{relevant}]{1/g_0^2>0} \text{ gauge-theory regime,} \tag{I.7.68}$$

so switching $1/g_0^2$ on moves away from the fixed point. A Coulomb vacuum additionally requires $\langle\phi\rangle \neq 0$. Sending the deformation to zero returns to the superconformal theory with its enhanced E_{N_f+1} symmetry.

These theories are strongly coupled at the fixed point and have no known weakly coupled Lagrangian description. The gauge Lagrangian describes a mass deformation away from that point. The exceptional symmetry E_{N_f+1} is invisible in that Lagrangian,

$$\text{Lagrangian shows } SO(2N_f)\times U(1)_I, \qquad \text{fixed point has } E_{N_f+1}\supset SO(2N_f)\times U(1)_I, \tag{I.7.69}$$

The perturbative description sees only the subgroup; instanton currents supply the missing generators at the fixed point.

Common misconception: a $5d$ theory without a Lagrangian is not a real theory

For the $SU(2)+N_f$ families in this chapter, the non-renormalizable gauge theory flows from an E_n superconformal fixed point with no known weakly coupled Lagrangian description; the gauge theory is a relevant (mass) deformation of the fixed point, not the other way around. These $5d$ superconformal theories are genuine quantum field theories. They have well-defined Coulomb-branch prepotentials, instanton operators, exceptional flavor symmetries, and superconformal indices, and their observables can be computed, even though no weakly-coupled Lagrangian exists. A missing Lagrangian is not a missing theory. This continues the non-Lagrangian boundary named in Chapter 2 and echoes the four-dimensional Argyres–Douglas theories of Chapter 5: the absence of a Lagrangian is a feature of many strongly-coupled fixed points, not a defect.

The convexity of the prepotential is a local field-theory test of the proposed fixed point. For $\phi \geq 0$ and a strictly positive inverse bare coupling, metric positivity (I.7.35) bounds the allowed matter,

$$\tau(\phi) > 0 \text{ for all } \phi \geq 0, \quad 1/g_0^2 > 0 \quad\Longleftrightarrow\quad 8 - N_f \geq 0 \quad (N_f \leq 8). \tag{I.7.70}$$

At $1/g_0^2 = 0$, $N_f = 8$ gives $\tau = 0$, so strict positivity fails even though the asymptotic slope is nonnegative. At $N_f = 7$ in the same limit, $\tau = \phi > 0$ for $\phi > 0$ and vanishes at the chamber origin. The ordinary E_{N_f+1} SCFT window is the strict subwindow $8 - N_f > 0$, $N_f \leq 7$ (with $N_f = 7$ giving E_8), while $N_f = 8$ ($8 - N_f = 0$) is the affine-E_8 / $6d$-lift boundary, not an ordinary $5d$ E_9 fixed point. The prepotential does not prove the fixed point exists, but it detects the obstruction when the theory is too heavily matter-loaded to have one. The sequence runs E_{N_f+1} for $N_f = 1, \ldots, 7$, up to E_8 at $N_f = 7$; the existence and classification are developed in Part III and Volume X.

The $N_f = 0$ caveat. Pure $SU(2)$ is not a single row. Its instanton sectors are classified by a $\mathbb{Z}_2$,

$$\pi_4(SU(2)) \;=\; \mathbb{Z}_2 \quad\Longrightarrow\quad \theta \in \{0, \pi\}, \tag{I.7.71}$$

a discrete theta angle. The two choices give different theories,

$$\theta = 0 :\ U(1)_I \to SU(2)_I\ (E_1), \qquad \theta = \pi :\ \text{only } U(1)_I\ (\widetilde{E}_1), \tag{I.7.72}$$

with a further mass deformation reaching the rank-zero E_0. Conventions for the lower $E_0/E_1/E_2$ rows vary, so we keep $N_f = 0$ compact as this discrete-theta pair and lean on $N_f = 5, 6, 7 \to E_6, E_7, E_8$ as the unambiguous anchors.

No $5d$ $\mathcal{N} = 2$ superconformal algebra. The Nahm classification of Chapter 1 permits a $5d$ superconformal algebra for exactly one amount of supersymmetry,

$$5d \text{ SCA} :\ F(4), \quad \mathcal{N} = 1\ (8\,\mathcal{Q}), \tag{I.7.73}$$

the exceptional superalgebra $F(4)$. The $5d$ $\mathcal{N} = 2$ theory has twice as many supercharges, above the ceiling,

$$5d\ \mathcal{N} = 2 = 16\,\mathcal{Q}\ >\ 8\,\mathcal{Q} \quad \Longrightarrow \quad \text{no interacting } 5d\ \mathcal{N} = 2 \text{ SCFT}, \tag{I.7.74}$$

so maximal $5d$ super Yang–Mills has no fixed point of its own; its completion is the $6d$ $(2, 0)$ theory on a circle, developed in the next chapter. We state this boundary fact and recall the classification from Chapter 1; this supercharge count explains the boundary of the five-dimensional case.

Geometric realizations

The field-theory quantities above also have geometric realizations. The following correspondences provide a picture of the fixed-point physics; their construction is developed in later volumes. Two constructions realize the $5d$ superconformal theories: the (p, q) five-brane web in type IIB string theory, and geometric engineering by M-theory on a local Calabi–Yau threefold. In both, the three field-theory objects of this chapter become geometry,

$$\begin{aligned} \text{Coulomb metric / prepotential } \mathcal{F} &\longleftrightarrow \text{ web face areas / CY triple intersection,} \\ \text{instanton particle} &\longleftrightarrow \text{ wrapped brane (M2 on a curve),} \\ \text{enhanced } E_{N_f+1} &\longleftrightarrow \text{ collapsed curves / 7-branes.} \end{aligned} \tag{I.7.75}$$

The web and Calabi–Yau pictures encode the same Coulomb metric, instanton particles, and enhanced flavor symmetry. Their geometric derivations belong to VII.7 and Volume X.

I.7.6 The $5d$ superconformal index

The enhanced E_{N_f+1} flavor symmetry of a $5d$ superconformal theory is not manifest in any Lagrangian, so one needs a protected observable to see it. That observable is the $5d$ superconformal index. It is a partition function on $S^4 \times S^1$, a weighted trace over the states of the theory on S^4,

$$\mathcal{I}(x, \{u_a\}) \ =\ \mathrm{Tr}_{\mathcal{H}(S^4)} (-1)^F e^{-\beta\Delta} x^{\,2(j_1+R)} \prod_a u_a^{F_a}, \tag{I.7.76}$$

Here $\Delta = \{Q, Q^\dagger\} = E - 2j_1 - 3R$ for the chosen supercharge. The Cartans j_1, j_2 belong to the S^4 rotation group, R is the $SU(2)_R$ Cartan, and $j_1 + R$ commutes with Q and $Q^\dagger$. We set the optional y^{2j_2} refinement to $y = 1$; u_a grade flavor and instanton charges F_a. The trace includes all states on S^4; states with $\Delta > 0$ cancel in pairs, leaving the BPS condition $\Delta = 0$, or $E = 2j_1 + 3R$. By radial quantization, states on S^4 correspond to local operators and the S^1 is the trace. The scalar in a conserved-current multiplet has $E = 3$, $R = 1$, $j_1 = j_2 = 0$, hence contributes x^2; grading it by $x^{2(E+j_1)}$ would instead give x^6. The protected trace is invariant under the stated deformation. In particular it is blind to $1/g_0^2$,

$$\frac{\partial \mathcal{I}}{\partial(1/g_0^2)} \;=\; 0 \quad\Longrightarrow\quad \mathcal{I}_{\text{gauge theory}} \;=\; \mathcal{I}_{\text{SCFT}}, \tag{I.7.77}$$

so the weakly-coupled gauge theory and its strongly-coupled fixed point share one index. A quantity computed in the gauge theory by summing over instanton sectors thus equals a quantity of the fixed point, which is how a gauge-theory computation sees the enhanced symmetry of a theory with no Lagrangian.

The enhancement is visible at the first nontrivial order. The coefficient of x^2 in the index, the conserved-current contribution, counts the conserved currents of the theory, refined by their flavor and instanton charges. Computed in the gauge theory it has two parts: the perturbative currents, which assemble the adjoint character of $SO(2N_f)\times U(1)_I$, and the one-instanton contribution, the $k = \pm 1$ sector, which supplies the spinor characters. Adding them,

$$\chi_{\text{adj}}(E_{N_f+1}) \;=\; \chi_{\text{adj}}(SO(2N_f)) \;+\; \chi_{U(1)_I} \;+\; \chi^{(k=+1)}_{\text{spinor}} \;+\; \chi^{(k=-1)}_{\text{spinor}}, \tag{I.7.78}$$

the conserved-current order reorganizes into the adjoint character of E_{N_f+1}. This simple form (the perturbative adjoint, the $U(1)_I$ current, and the $k = \pm 1$ spinor characters) closes the E_{N_f+1} adjoint through $N_f \le 5$; for $N_f = 6, 7$ it under-counts, and higher/dressed instanton sectors complete the E_7 and E_8 adjoints (§I.7.4), an enumeration developed in Part III. Setting all fugacities to one evaluates each character at the identity, giving the dimension identity, exactly the count (I.7.62) of §I.7.4 now from the index side,

$$\dim E_6 \;=\; \dim SO(10) + 1 + \dim \mathbf{16} + \dim \overline{\mathbf{16}} \;=\; 45 + 1 + 16 + 16 \;=\; 78. \tag{I.7.79}$$

The two routes agree. Dropping the instanton characters breaks it,

$$\dim SO(10) + 1 \;=\; 45 + 1 \;=\; 46 \;\neq\; 78, \tag{I.7.80}$$

so the instanton operators are as essential to the index as to the symmetry count.

There is no $5d$ $\mathcal{N} = 2$ superconformal algebra, so this index story is an $\mathcal{N} = 1$-only object: the superconformal index counts states annihilated by a supercharge of the $5d$ superconformal algebra $F(4)$, which exists only for eight supercharges. The full index machinery, the instanton-partition-function integrand that produces (I.7.76) order by order and the higher-order coefficients beyond the conserved-current order, is developed in Part III; here the relevant result is the conserved-current order and its E_{N_f+1} reorganization, worked at $N_f = 5$.

Exit checklist

After this chapter the reader can

1. identify the $5d$ $\mathcal{N}=1$ vector multiplet (gauge field, one real adjoint scalar ϕ, symplectic-Majorana gaugino, $SU(2)_R$) and hypermultiplet (four real scalars, Dirac fermion), read off the eight-supercharge count, and derive $[g^2]=4-d=-1$ from $[\frac{1}{g^2}\mathrm{Tr}\,F^2]=d$, so that $1/g_0^2$ is a dimensionful mass parameter, not a marginal coupling;

2. recognize the real Coulomb branch as a Weyl chamber of real dimension $\mathrm{rank}\,G$ (the half-line $\phi\geq 0$ for $SU(2)$), and distinguish it from the complex special-Kähler branch of $4d$ $\mathcal{N}=2$;

3. assemble the cubic prepotential $\mathcal{F}(\phi)=\frac{1}{2g_0^2}\phi^2+\frac{1}{12}(\sum_{\text{roots}}|\alpha\cdot\phi|^3-\sum_{\text{hypers}}|w\cdot\phi+m|^3)$ from the root/weight $|m|^3$ contributions, hold the instanton data separate from this classical plus one-loop cubic, differentiate the $SU(2)+N_f$ assembly to the leading cubic coefficient $8-N_f$ of $6\mathcal{F}$, read the metric $\tau(\phi)=\mathcal{F}''(\phi)=1/g_0^2+(8-N_f)\phi$ as a second derivative, and locate the sub-chamber walls at $\phi=|m_f|$;

4. verify metric positivity on the branch for $N_f\leq 8$ and the strict E_{N_f+1} window $N_f\leq 7$, and reject the doubled coefficient $8-2N_f$ that would forbid E_6, E_7, E_8;

5. define the topological $U(1)_I$ current $j_I=\frac{1}{8\pi^2}\mathrm{Tr}(F\wedge F)$ and the instanton operators as disorder operators of charge k on a surrounding S^4, anchor the instanton mass at $M_{\text{inst}}\sim 8\pi^2/g_0^2$ (dimension one, rejecting the dimension-two ϕ/g^2), and count the E_{N_f+1} flavor enhancement through $\dim E_6=\dim SO(10)+1+16+16=78$ with the branching $\mathbf{78}=\mathbf{45}_0+\mathbf{1}_0+\mathbf{16}_{+1}+\overline{\mathbf{16}}_{-1}$, remembering that E_{N_f+1} is a flavor symmetry, not the R-symmetry $SU(2)_R$;

6. explain why $1/g_0^2\to 0$ is a $5d$ superconformal fixed point with no Lagrangian description (the E_n theories), read the same enhancement off the conserved-current order of the $5d$ superconformal index on $S^4\times S^1$, and state the boundary fact that there is no interacting $5d$ $\mathcal{N}=2$ superconformal theory (sixteen supercharges sits above the eight-supercharge ceiling of the unique $5d$ superconformal algebra $F(4)$), with the fixed-point existence, the full instanton partition function, and the $6d$ origin to their later homes.

Status and sources

On the five-dimensional Coulomb branch, the vector-multiplet scalars are real. The eight-supercharge count and $SU(2)_R$ follow Chapter 1. The instanton operator is a disorder operator charged under the topological current. The fixed-point, current, index, multiplet and geometric identifications are imported physical inputs. The mathematical calculations apply under the stated hypotheses.

§I.7.1 Multiplets and the irrelevant coupling. The $5d$ $\mathcal{N}=1$ vector (one real adjoint scalar, $SU(2)_R$) and hypermultiplet (four real scalars, Dirac fermion), the eight-supercharge count, and the sign flip $[g^2]=4-d=-1$ (I.7.8) making $1/g_0^2$ a dimensionful mass parameter (the $U(1)_I$ real mass), contrasted with the marginal $4d$ value and with the $4d$ $\mathcal{N}=2$ complex adjoint of Chapter 5. (Seiberg (1996) the $5d$ fixed points and multiplet structure). Additional calculations include the 8Q count from the Chapter 1 table, the R-symmetry $SU(2)_R$ (rank 1, dim 3), $[g^2]=-1$ in $5d$ versus 0 in $4d$, and $[1/g_0^2]=+1$.

§I.7.2 The real Coulomb branch. The real adjoint vev breaking $G\to U(1)^r$ (I.7.11), the real Coulomb branch of dimension rank G (I.7.12) (the $SU(2)$ half-line $\phi\geq 0$ (I.7.13)), and the contrast with the complex special-Kähler branch of $4d$ $\mathcal{N}=2$ (Chapter 5, recovered on a circle by A_5 complexifying ϕ). (Seiberg (1996); Intriligator et al. (1997) the Coulomb-branch structure). Additional calculations include the real branch dimension = rank (the $SU(2)$ half-line with a single wall at $\phi=0$).

§I.7.3 The cubic prepotential and its metric. The two-piece prepotential (I.7.21) (classical $\frac{1}{2g_0^2}\phi^2$ plus the $SU(N\geq 3)$ Chern–Simons cubic, $SU(2)$ having none; the one-loop $|m|^3$ assembly (I.7.24), piecewise cubic), classical plus one-loop exact with the instanton data held separate; the metric as the second derivative $\tau_{ij}=\partial^2\mathcal{F}$ (I.7.26); the $SU(2)+N_f$ assembly (I.7.28) differentiated to the leading cubic coefficient $8-N_f$ of $6\mathcal{F}$ (I.7.31), the affine metric $\tau(\phi)=1/g_0^2+(8-N_f)\phi$ (I.7.33), positivity for $N_f\leq 8$ and the strict window $N_f\leq 7$ (I.7.35), and the chamber walls at $\phi=|m_f|$ (Table I.7.1). (Seiberg (1996); Intriligator et al. (1997) the full cubic prepotential and Coulomb-branch phases). Additional calculations include the one-loop $|m|^3$ assembly retaining the absolute values, the leading coefficient $8-N_f$ derived by differentiating with the chosen roots ± 2 / weights ± 1, the metric as $\mathcal{F}''$ (not $\mathcal{F}'$), the $N_f\leq 7$ window / $N_f=8$ boundary, and the walls $\{0,|m_f|\}$.

§I.7.4 Instanton operators and the E_n enhancement. The topological current $j_I=\frac{1}{8\pi^2}\mathrm{Tr}(F\wedge F)$ (I.7.49) and the integer instanton number (I.7.51), the instanton operators as disorder operators of charge k on a surrounding S^4 (I.7.52) (the $5d$ analog of the $3d$ monopole operator), the instanton-particle mass anchor $M_{\mathrm{inst}}\sim 8\pi^2/g_0^2$ (I.7.54) (dimension one), the perturbative $SO(2N_f)\times U(1)_I$ enhancing to E_{N_f+1} (I.7.58), and the worked $N_f=5\to E_6$ count $45+1+16+16=78$ (I.7.62) with the adjoint branching (I.7.63) (Table I.7.2; the naive two-spinor count honestly under-counting at $N_f=6,7$). (Lambert et al. (2015) the $5d$ instanton operators; Tachikawa (2015) the instanton-current symmetry enhancement; Seiberg (1996) the E_n fixed points). The current uses the stated prefactor $1/(8\pi^2)$ and is closed by the Bianchi identity. The worked example relates the basic operator's $U(1)_I$ charge 1, the mass anchor $8\pi^2/g_0^2$ of dimension 1 (the ϕ/g^2 form of dimension 2 rejected), $\dim SO(10)=45$ / $\dim\mathbf{16}=2^4$ / $\dim E_6=78$ from the E_6 root system, the identity $78=45+1+16+16$, and the naive count closing at $N_f\leq 5$ but under-counting at $N_f=6,7$ ($131<133$, $220<248$). The enhancement is of flavor symmetry.

§I.7.5 The $5d$ superconformal fixed points. The $1/g_0^2\to 0$ ultraviolet limit as a $5d$ superconformal theory, the gauge theory as a relevant (mass) deformation, the no-Lagrangian fact, the prepotential convexity as a necessary local test (nonnegative slope at $N_f\leq 8$, with the sourced ordinary fixed-point family ending at $N_f=7$), the $N_f=0$ discrete-θ pair $E_1/\widetilde{E}_1$ ($\pi_4(SU(2))=\mathbb{Z}_2$), and the boundary fact that there is no interacting $5d$ $\mathcal{N}=2$ superconformal theory (the unique $5d$ superconformal algebra is $F(4)$ at eight supercharges; maximal $5d$ SYM UV-completes to $6d$ $(2,0)$ on a circle, forward-pointed to Chapter 8). (statement-level; the existence/classification routed) (Seiberg (1996) the fixed-point existence; Córdova et al. (2016) the superconformal-algebra catalog / $F(4)$). Additional calculations include $5d$ $\mathcal{N}=2=16$Q

above the eight-supercharge $F(4)$ ceiling, the $5d$ SCA is $F(4)$ at $\mathcal{N}=1$, and the $\mathcal{N}=2$ fixed point forbidden. a non-Lagrangian $5d$ theory is a real theory.

§I.7.6 The $5d$ superconformal index. The index on $S^4 \times S^1$ (I.7.76), with $\Delta = E - 2j_1 - 3R$ and the commuting $x^{2(j_1+R)}$ grading at $y=1$, as a protected, deformation-invariant counting partition function, the enhancement visible at the conserved-current (x^2) order as the adjoint character of E_{N_f+1} (I.7.78), the index-side dimension identity $\dim E_6 = 45+1+16+16 = 78$ (I.7.79) agreeing with the symmetry count route, and the $\mathcal{N}=1$-only nature of the index (no $5d$ $\mathcal{N}=2$ superconformal algebra). (the full index machinery routed) (Kim et al. (2012), §3 eqs. (3.1)–(3.2) for the BPS condition and grading, and §4.2 for the x^2 current character; Tachikawa (2015), §2 eq. (2.1) for the dimension-three $SU(2)_R$-triplet scalar). Additional calculations include the index-side dimension sum equalling $\dim E_{N_f+1}$ computed from the E-series Cartan root system on the clean window $N_f = 4, 5$. The symmetry count and index character share the group-dimension input. Without the instanton characters, the example gives $46 \neq 78$.

Routed (stated, not proved). The existence and classification of the $5d$ superconformal fixed points (the E_n sequence, the higher-rank $5d$ theories, the convexity / gauge-theory-phase classification, the string / M-theory / brane-web construction that establishes existence) are routed to Part III and Volume X. The full $5d$ instanton-operator technology (the Nekrasov instanton partition function, the higher-order superconformal-index evaluation, the proof of the symmetry enhancement beyond the conserved-current count, and the brane-web realization of instanton particles) is routed to Part III and Volume X. No $5d$ holographic dual, S^5 free energy, or $5d$ central-charge technology is developed beyond naming it. The $6d$ origin of the $5d$ E_n theories on a circle, and the completion of maximal $5d$ SYM to $6d$ $(2,0)$, are forward-pointed to Chapter 8. The engine connection, the (p,q) five-brane webs and local-Calabi–Yau geometries that engineer $5d$ superconformal theories and realize the cubic prepotential, the instanton particles, and the E_n enhancement, is a later use in VII.7 and Volume X, not a claim here.

Further reading

Five-dimensional superconformal fixed points and their exceptional flavor symmetry are in Seiberg (1996); the geometric and extremal-transition picture in Morrison and Seiberg (1997) and Intriligator et al. (1997); the (p,q) five-brane webs in Aharony and Hanany (1997) and Aharony, Hanany, and Kol (1998). The systematic geometric classification of $5d$ superconformal theories is developed in Jefferson et al. (2018) and Closset et al. (2021).

Recent classification and compactification perspectives on $5d$ SCFTs are developed in Bhardwaj (2020) and Bhardwaj et al. (2020); the global form of flavor symmetry, instantonic symmetry, higher-form symmetry, and their anomalies are analyzed in Benetti Genolini and Tizzano (2021) and Apruzzi et al. (2022).

References

Aharony, O. and A. Hanany (1997). "Branes, superpotentials and superconformal fixed points". In: *Nucl. Phys. B* 504, pp. 239–271. DOI: 10.1016/S0550-3213(97)00472-0. arXiv: hep-th/9704170.

Aharony, O., A. Hanany, and B. Kol (1998). "Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams". In: *JHEP* 01, p. 002. DOI: 10.1088/1126-6708/1998/01/002. arXiv: hep-th/9710116.

Apruzzi, F. et al. (2022). "The Global Form of Flavor Symmetries and 2-Group Symmetries in 5d SCFTs". In: *SciPost Phys.* 13.2, p. 024. DOI: 10.21468/SciPostPhys.13.2.024. arXiv: 2105.08724 [hep-th].

Benetti Genolini, P. and L. Tizzano (2021). "Instantons, symmetries and anomalies in five dimensions". In: *JHEP* 04, p. 188. DOI: 10.1007/JHEP04(2021)188. arXiv: 2009.07873 [hep-th].

Bhardwaj, L. (2020). "On the classification of 5d SCFTs". In: *JHEP* 09, p. 007. DOI: 10.1007/JHEP09(2020)007. arXiv: 1909.09635 [hep-th].

Bhardwaj, L. et al. (2020). "Twisted Circle Compactifications of 6d SCFTs". In: *JHEP* 12, p. 151. DOI: 10.1007/JHEP12(2020)151. arXiv: 1909.11666 [hep-th].

Closset, C., S. Schafer-Nameki, and Y.-N. Wang (2021). "Coulomb and Higgs Branches from Canonical Singularities: Part 0". In: *JHEP* 02, p. 003. DOI: 10.1007/JHEP02(2021)003. arXiv: 2007.15600 [hep-th].

Córdova, C., T. T. Dumitrescu, and K. Intriligator (2016). "Deformations of Superconformal Theories". In: *JHEP* 11, p. 135. DOI: 10.1007/JHEP11(2016)135. arXiv: 1602.01217 [hep-th].

Intriligator, K. A., D. R. Morrison, and N. Seiberg (1997). "Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces". In: *Nucl. Phys. B* 497, pp. 56–100. DOI: 10.1016/S0550-3213(97)00279-4. arXiv: hep-th/9702198.

Jefferson, P. et al. (2018). "On Geometric Classification of 5d SCFTs". In: *JHEP* 04, p. 103. DOI: 10.1007/JHEP04(2018)103. arXiv: 1801.04036 [hep-th].

Kim, H.-C., S.-S. Kim, and K. Lee (2012). "5-dim Superconformal Index with Enhanced E_n Global Symmetry". In: *JHEP* 10, p. 142. DOI: 10.1007/JHEP10(2012)142. arXiv: 1206.6781 [hep-th].

Lambert, N., C. Papageorgakis, and M. Schmidt-Sommerfeld (2015). "Instanton Operators in Five-Dimensional Gauge Theories". In: *JHEP* 03, p. 019. DOI: 10.1007/JHEP03(2015)019. arXiv: 1412.2789 [hep-th].

Morrison, D. R. and N. Seiberg (1997). "Extremal transitions and five-dimensional supersymmetric field theories". In: *Nucl. Phys. B* 483, pp. 229–247. DOI: 10.1016/S0550-3213(96)00592-5. arXiv: hep-th/9609070.

Seiberg, N. (1996). "Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics". In: *Phys. Lett. B* 388, pp. 753–760. DOI: 10.1016/S0370-2693(96)01215-4. arXiv: hep-th/9608111.

Tachikawa, Y. (2015). "Instanton operators and symmetry enhancement in 5d supersymmetric gauge theories". In: *PTEP* 2015.4, 043B06. DOI: 10.1093/ptep/ptv040. arXiv: 1501.01031 [hep-th].

Chapter I.8

$6d$ supersymmetric field theories

Why six dimensions is different. The eight-supercharge theories of Chapters 5 and 7 have vector-multiplet scalars. In six dimensions the vector has no scalar; a chiral two-form instead comes with a scalar whose expectation value controls a gauge coupling and the tension of a string. At the point where the string becomes tensionless, the tensor-branch description breaks down. The same tensor supplies the Green–Schwarz coupling needed when a dynamical gauge anomaly has a factorized eight-form. This chapter follows that chain from the tensor's three physical polarizations through an explicit rank-one E-string anomaly calculation. It then uses the anomaly polynomial to read the cubic Euler anomaly of the interacting $\mathcal{N} = (2, 0)$ theory associated with coincident M5-branes.

What this chapter delivers

The $6d$ $\mathcal{N} = (1, 0)$ multiplets (vector, hyper, tensor with its anti-self-dual two-form) and the on-shell count that closes the tensor multiplet (§I.8.1); the tensor branch, the tensor scalar as the inverse gauge coupling, and the tensionless string at the origin (§I.8.2); the anomaly eight-form I_8, its $\hat{A}(R)\,\mathrm{ch}(F)$ blocks, and the irreducible $\mathrm{tr}\,F^4$ obstruction, worked on the rank-one E-string (§I.8.3); the Green–Schwarz–Sagnotti–West factorization $I_8 = \frac{1}{2}\Omega_{ij}X_4^i X_4^j$ and why the tensor branch is required (§I.8.4); the four Weyl coefficients (a, c_1, c_2, c_3), matched on the branch, with $6d$ beside the extremization family (§I.8.5); the $6d$ $\mathcal{N} = (2, 0)$ theory, its absence of a Lagrangian, the $(\mathbb{R}^5)^{N-1}/S_N$ branch, and the cubic $a \sim N^3$ (§I.8.6); and $6d$ $\mathcal{N} = (1, 1)$ maximal super-Yang–Mills as the Lagrangian contrast (§I.8.7).

The $6d$ supersymmetry data, recalled once. Six-dimensional supersymmetry comes in three amounts, and Chapter 1's master table fixed each. A minimal $6d$ chiral (Weyl) spinor supercharge carries eight real components, so the three cases are

$$\mathcal{N} = (1, 0) :\ 8, \qquad \mathcal{N} = (2, 0) :\ 16, \qquad \mathcal{N} = (1, 1) :\ 16 \tag{I.8.1}$$

real supercharges. The minimal case $6d$ $\mathcal{N} = (1, 0)$ has R-symmetry $SU(2)_R$ and sits

at the top of the eight-supercharge sequence,

$$6d\ \mathcal{N}=(1,0)\ \equiv\ 5d\ \mathcal{N}=1\ \equiv\ 4d\ \mathcal{N}=2\ \equiv\ 3d\ \mathcal{N}=4 \qquad \text{(8 supercharges)}, \quad \text{(I.8.2)}$$

the sibling family of Chapters 5 and 7. The two maximal cases each add a second supercharge; their chirality choice fixes the R-symmetry,

$$\begin{aligned} &\mathcal{N}=(1,0): SU(2)_R, \\ &\mathcal{N}=(2,0): USp(4)_R = Spin(5)_R, \\ &\mathcal{N}=(1,1): Spin(4)_R = SU(2)^2. \end{aligned} \qquad \text{(I.8.3)}$$

The $\mathcal{N}=(2,0)$ pair has the same chirality (larger R-symmetry, $SO(5)$); the $\mathcal{N}=(1,1)$ pair has opposite chirality ($SO(4)$). These are the Chapter 1 counts and R-symmetries. Chapter 2's distinction between a gauge anomaly, which must cancel, and a background 't Hooft anomaly, which may remain, will matter when we reach the E_8 flavor symmetry.

Part I. The $6d\ \mathcal{N}=(1,0)$ world

The first five sections build $6d\ \mathcal{N}=(1,0)$ dynamics: the multiplets and the chiral tensor, the tensor branch, the anomaly polynomial, its Green–Schwarz factorization, and the conformal anomalies.

I.8.1 Multiplets and the self-dual tensor

A $6d\ \mathcal{N}=(1,0)$ theory is assembled from three multiplets, the third with no lower-dimensional analog.

- *Vector*: a gauge field A_μ and a gaugino, with *no scalar* (unlike $4d/5d$, no adjoint Coulomb-branch scalar in $6d$).
- *Hyper*: four real scalars (an $SU(2)_R$ doublet, as in four and five dimensions) and a fermion, the $6d$ matter multiplet.
- *Tensor*: one real scalar ϕ, an antisymmetric two-form $B^-_{\mu\nu}$ with anti-self-dual $H^- = dB^-$, and a fermion. Its chiral two-form is what makes six dimensions special.

The vector and hyper are familiar; the tensor is the new object. Table I.8.1 collects all three alongside the maximal cases.

Six is the lowest dimension hosting a chiral two-form. The field strength $H = dB$ is a three-form, and in $d=6$ a three-form is middle-dimensional, $\star : \Omega^3 \to \Omega^{6-3} = \Omega^3$, so $H = \pm \star H$ makes sense. We take the $\mathcal{N}=(1,0)$ two-form anti-self-dual and the $\mathcal{N}=(2,0)$ (and supergravity) tensor self-dual,

$$\mathcal{N}=(1,0):\ H^- = -\star H^-, \qquad \mathcal{N}=(2,0):\ H^+ = +\star H^+. \qquad \text{(I.8.4)}$$

Only the sign is a convention; the physical content is that the two-form is chiral. We fix the $\mathcal{N}=(1,0)$ sign as in (I.8.4) so that the tensor-branch gauge kinetic term $\phi\,\mathrm{tr}\,F^2$ of §I.8.2 comes out with the right sign.

The R-symmetry $SU(2)_R$ organizes the fermions. The gaugino and the tensor fermion are $SU(2)_R$ doublets (each a symplectic-Majorana–Weyl spinor), the bosons A_μ, $B_{\mu\nu}$, ϕ are singlets, and only the hypermultiplet has $SU(2)_R$-charged bosons,

$$\text{doublets: } \lambda,\ \psi_T;\quad \text{singlets: } A_\mu,\ B_{\mu\nu},\ \phi;\quad \text{hyper: } (\mathbf{2},\mathbf{2}) \text{ of } SU(2)_R\times SU(2)_F. \tag{I.8.5}$$

The four hyper scalars form two $SU(2)_R$ doublets (a half-hypermultiplet is one doublet, for a pseudoreal representation), the same 8-supercharge state count as the $4d$ $\mathcal{N}=2$ and $5d$ $\mathcal{N}=1$ hypermultiplets of Chapters 5 and 7. The new element is the tensor multiplet.

That $H=\pm\star H$ is a real condition, not a vacuous one, rests on a signature check. The square of the Hodge star on a k-form in d Lorentzian dimensions is $(-1)^{k(d-k)}s$ with $s=-1$; on a $6d$ three-form,

$$\star^2\big|_{k=3,\,d=6} \;=\; (-1)^{k(d-k)}\,s \;=\; (-1)^{3\cdot 3}\,(-1) \;=\; (-1)^9\,(-1) \;=\; +1. \tag{I.8.6}$$

Because $\star^2=+1$ the projectors $\frac{1}{2}(1\pm\star)$ are genuine and $H=\pm\star H$ has nontrivial real solutions. In Euclidean signature $s=+1$ gives $\star^2=-1$, the real condition collapses, and the chiral tensor is a Lorentzian object. This signature result is what permits real chiral tensors in Lorentzian six dimensions.

Worked instance: the tensor-multiplet degree-of-freedom count

The tensor multiplet must close the supersymmetry algebra, so its on-shell bosonic and fermionic degrees of freedom must match. Counting them is the smallest piece of tensor-multiplet arithmetic, and it feeds directly into the anomaly polynomial of §I.8.3. Start with the two-form off shell: an antisymmetric $B_{\mu\nu}$ in six dimensions has

$$\binom{6}{2} \;=\; 15 \tag{I.8.7}$$

independent components. Gauge redundancy and the equations of motion cut this to physical states, counted cleanest in the massless little group $SO(4)=SU(2)\times SU(2)$. The three-form field strength splits self-dual and anti-self-dual, sitting in the $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$, so a generic two-form carries both while self-duality keeps one half,

$$(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{3}) = 3+3=6 \;\xrightarrow{\;H^-=-\star H^-\;}\; (\mathbf{1},\mathbf{3})=3. \tag{I.8.8}$$

The fermion is symplectic-Majorana–Weyl. After imposing its reality condition, it has eight real components off shell; the massless Dirac equation halves these to four on-shell states,

$$n_{\text{fermi}} \;=\; \tfrac{1}{2}\cdot 8 \;=\; 4. \tag{I.8.9}$$

Adding the real scalar ϕ to the tensor's 3 states, the counts balance,

$$n_{\text{bose}} \;=\; \underbrace{3}_{B^-} + \underbrace{1}_{\phi} \;=\; 4 \;=\; \underbrace{4}_{\psi_T} \;=\; n_{\text{fermi}}, \tag{I.8.10}$$

as required for an eight-supercharge tensor multiplet. This checks its state count; closure of the supersymmetry transformations is additional structure.

The same counting fixes the other two multiplets, so the anomaly assembly of §I.8.3 has all three. The vector has a gauge field ($6-2=4$ states, an $SO(4)$ vector) plus a gaugino; the hyper has four real scalars plus a Weyl fermion:

$$\text{vector}: \ 4_B = 4_F, \qquad \text{hyper}: \ 4_B = 4_F, \qquad \text{tensor}: \ (3+1)_B = 4_F. \tag{I.8.11}$$

Each $\mathcal{N}=(1,0)$ multiplet carries $(4,4)$ on-shell states, the smallest 8-supercharge representations; the tensor is the one whose bosonic 4 splits $3+1$ rather than $4+0$, and that split feeds the self-dual-tensor term into the anomaly polynomial. The three states are a chiral two-form, the origin of every $6d$-specific structure below: no covariant Lagrangian (§I.8.6), a self-dual string (§I.8.2), a Hirzebruch term in I_8 (§I.8.3).

supersymmetry	supercharges	R-symmetry	multiplets
$6d$ $\mathcal{N}=(1,0)$	8	$SU(2)_R$	vector; hyper; tensor (anti-self-dual B^-)
$6d$ $\mathcal{N}=(2,0)$	16	$USp(4)_R$	tensor (self-dual B^+, five scalars)
$6d$ $\mathcal{N}=(1,1)$	16	$Spin(4)_R$	vector, four scalars: maximal SYM

Table I.8.1: The $6d$ supersymmetries, their real supercharge counts, R-symmetries (the Chapter 1 master-table $6d$ row), and multiplet content. The R-symmetries are $SU(2)_R$ for $\mathcal{N}=(1,0)$, $USp(4)_R = Spin(5)_R$ for $\mathcal{N}=(2,0)$, and $Spin(4)_R = SU(2)\times SU(2)$ for $\mathcal{N}=(1,1)$. The $\mathcal{N}=(1,0)$ vector has no scalar; the tensor multiplet's two-form is chiral. The $\mathcal{N}=(2,0)$ tensor is self-dual with five $SO(5)_R$ scalars; $\mathcal{N}=(1,1)$ maximal SYM is a Lagrangian gauge theory with four scalars. The supercharge counts agree with the Chapter 1 sequence.

I.8.2 The tensor branch

A $6d$ $\mathcal{N}=(1,0)$ theory with n_T tensor multiplets has n_T real tensor scalars ϕ^i, whose vacuum expectation values parametrize the *tensor branch*, the $6d$ member of Chapter 2's branch vocabulary. Its real dimension is exactly the number of tensor multiplets,

$$\dim_{\mathbb{R}} \mathcal{M}_{\text{tensor}} \ = \ n_T, \tag{I.8.12}$$

for a branch with n_T independent tensor scalars. The interacting SCFT sits at the origin $\langle\phi^i\rangle = 0$; generic points admit a low-energy description in tensor and, when present, gauge fields.

The tensor scalars set gauge kinetic coefficients. For one gauge factor coupled to one tensor, the relation takes the simple form

$$\mathcal{L}_{\text{gauge}} \supset \phi^i \operatorname{tr} F_i^2, \qquad \frac{1}{g_i^2} \sim \langle \phi^i \rangle, \qquad \langle \phi^i \rangle \to \infty : \; g_i \to 0, \quad \langle \phi^i \rangle \to 0 : \; g_i \to \infty. \tag{I.8.13}$$

For several tensors or gauge factors, the coefficient is an appropriate linear combination of the ϕ^i. Moving toward the origin sends the effective coupling to strong coupling along the indicated ray. Thus the tensor branch carries coupling data despite the absence of a vector-multiplet scalar.

Charged under B^i is a *self-dual string*, a one-dimensional soliton whose tension is the tensor scalar, tensionless at the origin,

$$T_i \sim \langle \phi^i \rangle \xrightarrow{\langle \phi^i \rangle \to 0} 0. \tag{I.8.14}$$

A tensionless string is a new kind of light degree of freedom, not a particle, and its appearance is the hallmark of an interacting $6d$ fixed point: the origin contains light string excitations no Lagrangian in the tensor-branch fields can capture. It is the $6d$ counterpart of a massless BPS particle on a lower-dimensional Coulomb branch, one dimension bigger. The vanishing follows from the stated proportionality; the microscopic string dynamics enters later constructions.

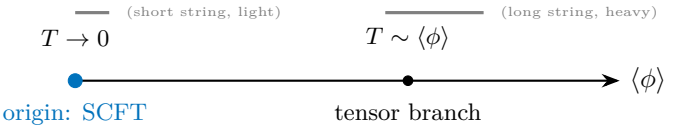


Figure I.8.1: The $6d$ $\mathcal{N} = (1,0)$ tensor branch, schematic. The real tensor scalar $\langle \phi \rangle$ runs outward from the origin. On the branch the gauge coupling is set by $1/g^2 \sim \langle \phi \rangle$ and a self-dual string (charged under B) has tension $T \sim \langle \phi \rangle$. At the origin (blue) the string becomes tensionless and the interacting superconformal field theory appears. The figure is a one-dimensional cartoon of an n_T-dimensional branch; the string, not a particle, is the light object at the origin.

Common misconception: the tensor branch is the Coulomb branch under another name

It is tempting to call the $6d$ tensor branch a Coulomb branch, since both are moduli spaces carrying the low-energy coupling data, but the difference is structural. The $4d/5d$ Coulomb branch is parametrized by *vector-multiplet* scalars, and the $6d$ $\mathcal{N} = (1,0)$ vector has *no scalar at all*, so there is no $6d$ Coulomb branch of that type. The moduli space carrying the special $6d$ data is parametrized by *tensor-multiplet* scalars, and it is the tensor branch. Branch names attach to the theory's own multiplets (Chapter 2); on the tensor branch the scalar plays the inverse-coupling role (I.8.13) and a self-dual string, not a particle, goes light at the origin.

A purely gravitational constraint ties n_T to the spectrum. The pure-gravitational anomaly must itself be Green–Schwarz-removable, which coupled to gravity becomes

the $6d$ gravitational-anomaly condition

$$n_H \;-\; n_V \;+\; 29\,n_T \;=\; 273, \tag{I.8.15}$$

one of the sharpest constraints on $6d$ spectra and the reason n_T is not free; the coupling to gravity is developed in later volumes.

The tensor branch is where the anomaly technology lives. The origin SCFT has an I_8 hard to compute directly; on the branch the theory is free, so its I_8 is a sum of free-field blocks. Since I_8 is an RG invariant, the matching

$$I_8^{\text{SCFT}} \;=\; I_8^{\text{massless branch}} + I_8^{\text{GS}} \tag{I.8.16}$$

makes it computable: the Green–Schwarz term accounts for the mismatch between the origin and the massless branch fields. This relation connects factorization below to the conformal anomalies. Later F-theory constructions supply particular tensor branches.

I.8.3 The anomaly polynomial

A chiral theory in six dimensions has gauge, gravitational, mixed, and R-symmetry anomalies, and the right object to organize them is a polynomial. The anomaly-inflow formalism packages the anomaly of a d-dimensional theory into a characteristic class of degree $d+2$; a one-loop anomaly in $2n$ dimensions comes from an $(n+1)$-gon diagram, so

$$d=2:\; I_4 \text{ (2-gon)}, \qquad d=4:\; I_6 \text{ (triangle)}, \qquad d=6:\; I_8 \text{ (box)}, \tag{I.8.17}$$

and in six dimensions the whole analysis is arithmetic on the eight-form I_8, generated by a box (four-point) diagram. The anomaly is recovered from I_8 by the Stora–Zumino descent chain. Because I_8 is a closed, gauge-invariant characteristic class it is locally exact, and one descends by two steps,

$$I_8 \;=\; dI_7^{(0)}, \qquad \delta_\lambda I_7^{(0)} \;=\; dI_6^{(1)}, \qquad \delta_\lambda \log Z \;=\; 2\pi i \int_{M_6} I_6^{(1)}, \tag{I.8.18}$$

the Chern–Simons seven-form $I_7^{(0)}$, then its gauge variation (λ the gauge/Lorentz parameter) descending to the six-form $I_6^{(1)}$, and finally $\int_{M_6} I_6^{(1)}$ as the anomalous variation of the effective action. Each step lowers the degree by one, $8 \to 7 \to 6$, landing on a six-form over spacetime. The entire anomaly is packaged in the single polynomial I_8, and the physics questions become algebra on its coefficients.

Common misconception: the $6d$ anomaly is a triangle diagram

In four dimensions the anomaly comes from a triangle (three-point) diagram and the polynomial is the six-form I_6. Six dimensions is not the same: a d-dimensional anomaly is a degree-$(d+2)$ class, so the $6d$ anomaly is the *eight-form* I_8, generated by a *box* (four-point) diagram, per the $2n$-dimensional $(n+1)$-gon pattern of (I.8.17). The consequence is that $6d$ cancellation is richer than the $4d$ cubic vanishing: an irreducible $\operatorname{tr} F^4$ piece must cancel group-theoretically *and* the residual must factorize for Green–Schwarz inflow to absorb it. The $6d$ tool is the eight-form polynomial and its factorization, not a single triangle.

The building blocks. The anomaly polynomial is assembled additively from the chiral field content, each field contributing with its chirality sign. A chiral Weyl fermion in a representation R of the gauge or flavor group contributes the degree-eight part of the index-density form

$$\big[\hat{A}(R)\,\mathrm{ch}(F)\big]_8, \tag{I.8.19}$$

the entry-9 inflow building block, $\hat{A}(R)$ the Dirac genus and $\mathrm{ch}(F)$ the Chern character, the subscript 8 selecting the eight-form; a chiral tensor contributes a signature (Hirzebruch) term. Expanding the two factors,

$$\hat{A} \;=\; 1 \;-\; \frac{p_1}{24} \;+\; \frac{7p_1^2 - 4p_2}{5760} \;+\; \ldots, \qquad \mathrm{ch}(F) \;=\; \mathrm{rk} \;+\; \mathrm{tr}\, e^{iF/2\pi}\big|_{\geq 2}, \tag{I.8.20}$$

the degree-eight product collects gravitational, mixed, and gauge pieces,

$$[\hat{A}\,\mathrm{ch}]_8 \;\supset\; \underbrace{p_1^2,\, p_2}_{\hat{A}|_8}, \quad \underbrace{(\mathrm{tr}\, F^2)\, p_1}_{\text{deg-4}\,\times\,\text{deg-4}}, \quad \underbrace{\mathrm{tr}\, F^4}_{\mathrm{ch}|_8}, \tag{I.8.21}$$

plus the $c_2(R)$ monomials from the same expansion on the $SU(2)_R$ background. Products such as $(\mathrm{tr}\, F^2)^2$ arise when a residual anomaly factorizes or when a group trace reduces; they do not occur as an independent term in the single-fermion Chern character. The index-theorem derivation belongs to the later anomaly treatment. The full degree-eight basis, built from F, p_1, p_2, and $c_2(R)$, is

$$\mathrm{tr}\, F^4, \quad (\mathrm{tr}\, F^2)^2, \quad (\mathrm{tr}\, F^2)\, p_1, \quad p_1^2, \quad p_2, \quad c_2(R)^2, \quad c_2(R)\, p_1, \quad c_2(R)\, (\mathrm{tr}\, F^2), \tag{I.8.22}$$

and assembling I_8 means computing each rational coefficient from the field content.

The gauge dependence enters through group-theory index data. For a chiral fermion in a representation R, the quartic invariants reduce to a reference (the fundamental, or the smallest exceptional representation) by

$$\mathrm{tr}_R F^2 \;=\; A_R\, \mathrm{tr}\, F^2, \qquad \mathrm{tr}_R F^4 \;=\; B_R\, \mathrm{tr}\, F^4 \;+\; C_R\, (\mathrm{tr}\, F^2)^2, \tag{I.8.23}$$

with A_R the Dynkin index and B_R, C_R the quartic index data of R. The coefficient of the irreducible $\mathrm{tr}\, F^4$ in I_8 is then a signed sum over the chiral matter, the gaugino minus the hypermultiplets,

$$\big[I_8\big]_{\mathrm{tr}\, F^4} \;\propto\; B_{\mathrm{adj}} \;-\; \sum_f n_f\, B_{R_f}, \tag{I.8.24}$$

while the reducible $(\mathrm{tr}\, F^2)^2$ collects the C_R and A_R pieces. When the group has no independent quartic Casimir, $\mathrm{tr}\, F^4$ itself reduces to $(\mathrm{tr}\, F^2)^2$ and $B_R \to 0$; the analysis splits into a B_R (irreducible) part that must vanish and a C_R/A_R (reducible) part that must factorize.

The irreducible $\mathrm{tr}\, F^4$ obstruction. The single-trace quartic $\mathrm{tr}\, F^4$ is *irreducible*: it is not a product of two degree-four forms, so it has no home in the factorized $\frac{1}{2}\Omega_{ij} X_4^i X_4^j$ (a sum of squares) and Green–Schwarz inflow cannot absorb it. Consistency therefore

forces its coefficient to *vanish on its own*, from the group theory of the matter, the six-dimensional analog of the $4d$ cubic gauge anomaly. Whether a group carries an independent quartic Casimir decides the story,

$$\text{has one: } SU(N\geq 4),\ SO(N\geq 8),\ Sp(N);\quad \text{none: } SU(2), SU(3), G_2, F_4, E_6, E_7, E_8. \tag{I.8.25}$$

For the second list $\operatorname{tr} F^4$ reduces to $(\operatorname{tr} F^2)^2$. The sharpest case is E_8, where

$$\operatorname{tr}_{\mathbf{248}} F^4 \;=\; \frac{1}{100}\left(\operatorname{tr}_{\mathbf{248}} F^2\right)^2, \tag{I.8.26}$$

so an E_8 theory has no irreducible quartic at all: any would-be $\operatorname{tr} F^4$ folds, with the constant $\frac{1}{100}$, into the reducible $(\operatorname{tr} F^2)^2$ slot. For $SU(5)$, which has an independent quartic Casimir, this reduction is unavailable; its irreducible coefficient must cancel through the matter content.

Worked instance: the anomaly eight-form of the rank-one E-string

The example carried through this section and the next is the *rank-one E-string* theory: one tensor multiplet ($n_T = 1$), no gauge algebra on the tensor branch, and an E_8 flavor symmetry. Its anomaly-polynomial coefficients are the modern values of Ohmori, Shimizu, Tachikawa, and Yonekura (OSTY), the standard $6d$ Green–Schwarz source. The E-string is the cleanest genuine example: a single tensor, a single exceptional flavor group, and a residual anomaly that factorizes with one degree-four form.

First the group theory. Because the flavor group is E_8, the E_8 identity (I.8.26) applies, and the irreducible $\operatorname{tr} F_{E_8}^4$ coefficient in the assembled I_8 is

$$\big[I_8\big]_{\operatorname{tr} F^4} \;=\; 0. \tag{I.8.27}$$

There is no independent quartic Casimir for E_8 to obstruct the factorization; the whole E_8 dependence collapses into $\operatorname{Tr} F_{E_8}^2$ and its square. Here E_8 is a *background flavor* symmetry. Its terms in I_8 are 't Hooft anomaly data, including any flavor dependence left after the group identity. Vanishing of an irreducible term is a consistency requirement for a *dynamical* gauge field with such a quartic invariant; this E_8 calculation instead illustrates the algebra of the reducible flavor-dependent residual.

We distinguish three objects: the rank-Q system including its decoupled center-of-mass hypermultiplet, the interacting rank-one theory, and the residual left after subtracting the free tensor on its tensor branch. Here $Q = 1$ counts one positive E-string rank, not a rank-zero relative A-series theory. Use the normalized trace $\operatorname{Tr} = \operatorname{tr}_{\text{adj}}/h^\vee$, with $h^\vee(E_8) = 30$, and include $1/(2\pi)$ in the anti-Hermitian field strength. Thus $c_2(R) = \operatorname{Tr} F_R^2/4$ is the second Chern class of the rank-two $SU(2)_R$ bundle, and $f = \operatorname{Tr} F_{E_8}^2$ below. This Tr differs from the unnormalized adjoint trace in (I.8.26).

The full source polynomial can be reduced before trying to factorize it. Writing $r = c_2(R)$, $l = c_2(L)$, $p = p_1(T)$ and $t = p_2(T)$, its ingredients are

$$p_1(N) = -2(l+r), \qquad \chi_4(N) = l - r, \qquad p_2(N) = (l-r)^2,$$

$$J_4 = \frac{p_1(N) + p + f}{4}, \qquad J_8 = \frac{p_2(N) + t - \frac{1}{4}(p_1(N) - p)^2}{48}.$$

$$I_8^{E+H}(Q) = \frac{Q^3 p_2(N)}{6} + \frac{Q^2 \chi_4(N) J_4}{2} + Q\left(\frac{J_4^2}{2} - J_8\right).$$

The center-of-mass hyper is a half-hyper in the $SU(2)_L$ doublet. Its polynomial and that of one free $(1,0)$ tensor are

$$I_8^H = \frac{7p^2 - 4t}{5760} + \frac{lp}{48} + \frac{l^2}{24}, \qquad I_8^T = \frac{r^2}{24} + \frac{rp}{48} + \frac{23p^2 - 116t}{5760}.$$

At $Q = 1$, subtracting I_8^H cancels every l term. The interacting polynomial is

$$I_8^E = \frac{13r^2}{24} - \frac{11rp}{48} - \frac{rf}{4} + \frac{203p^2 - 116t}{5760} + \frac{pf}{16} + \frac{f^2}{32}.$$

For example, its r^2 coefficient is $1/6 + 1/4 + 1/8 = 13/24$, directly from the three terms of $I_8^{E+H}(1)$. Subtracting I_8^T then gives $13/24 - 1/24 = 1/2$ in that slot and $-11/48 - 1/48 = -1/4$ in the rp slot. The rf coefficient remains $-1/4$. All $p_2(T)$ terms cancel in this subtraction; no separate gravitational piece is discarded. The resulting residual is

$$I_8^{\text{res}} \;=\; \tfrac{1}{2}\, X_4\, X_4, \qquad X_4 \;=\; -\tfrac{1}{4}\, p_1 \;-\; \tfrac{1}{4}\, \operatorname{Tr} F_{E_8}^2 \;+\; c_2(R), \tag{I.8.28}$$

one tensor ($\Omega = 1$) with a single degree-four X_4. This is the rank-one tensor-branch anomaly difference, in agreement with the direct OSTY factorization. The global sign $X_4 \mapsto -X_4$ leaves the residual unchanged; we choose the positive $c_2(R)$ coefficient. Expanding this square now checks it against the independently subtracted polynomial above,

$$\begin{aligned} I_8^{\text{res}} = {} & \tfrac{1}{32} p_1^2 + \tfrac{1}{32} (\operatorname{Tr} F_{E_8}^2)^2 + \tfrac{1}{2} c_2(R)^2 \\ & + \tfrac{1}{16} (\operatorname{Tr} F_{E_8}^2) p_1 - \tfrac{1}{4} c_2(R) p_1 - \tfrac{1}{4} c_2(R) (\operatorname{Tr} F_{E_8}^2). \end{aligned} \tag{I.8.29}$$

where each square coefficient is $\frac{1}{2}$ times the square of an X_4 coefficient and each cross coefficient is a product,

$$\tfrac{1}{2}(-\tfrac{1}{4})^2 = \tfrac{1}{32}, \quad \tfrac{1}{2}(1)^2 = \tfrac{1}{2}; \qquad (-\tfrac{1}{4})(-\tfrac{1}{4}) = \tfrac{1}{16}, \quad (1)(-\tfrac{1}{4}) = -\tfrac{1}{4}. \tag{I.8.30}$$

Every coefficient in (I.8.29) follows by expanding the stated X_4; Table I.8.2 records their origin as squares and products of the X_4 data. The cross coefficients test the square: they are fixed *products* of the X_4 coefficients, so an I_8 whose cross terms miss those products is not a perfect square. That conspiracy is Green–Schwarz factorization, the next section.

The gauge-anomaly cancellation condition

In a $6d$ $\mathcal{N} = (1,0)$ theory *with* a gauge group $\mathfrak{g}$ carrying an independent quartic Casimir, the $\operatorname{tr} F^4$ vanishing (I.8.24) becomes a constraint on the charged matter, the

working tool the reader most often needs. With hypermultiplets in representations R_f of multiplicity n_f,

$$B_{\rm adj} \;-\; \sum_f n_f\, B_{R_f} \;=\; 0 \tag{I.8.31}$$

must hold identically (B_R the quartic index of (I.8.23), hypers entering opposite to the gaugino), the direct $6d$ analog of the $4d$ cubic-anomaly cancellation. When $\mathfrak{g}$ has no independent quartic Casimir, (I.8.31) is vacuous and the whole gauge anomaly is left to Green–Schwarz. When it does hold, the residual reducible anomaly (A_R, C_R, gravitational, R-symmetry) must still factorize, fixing n_T and the lattice Ω_{ij}. Both steps are necessary; the independent $SU(5)$ quartic illustrates why (I.8.31) is a genuine constraint.

A concrete count makes this tangible. For $\mathfrak{su}(N)$ ($N \geq 4$), with the fundamental normalized to $B_\square = 1$, the quartic indices are

$$B_{\rm adj} \;=\; 2N, \qquad B_\square \;=\; 1, \qquad B_{\rm antisym} \;=\; N-8, \tag{I.8.32}$$

so with $n_\square$ fundamental and $n_{\rm a}$ antisymmetric hypermultiplets (I.8.31) reads

$$2N \;-\; n_\square \;-\; n_{\rm a}\,(N-8) \;=\; 0, \tag{I.8.33}$$

a Diophantine constraint on the matter with two standard solutions,

$$n_{\rm a}=0:\; n_\square = 2N; \qquad n_{\rm a}=1:\; n_\square = 2N-(N-8) = N+8. \tag{I.8.34}$$

The first, $\mathfrak{su}(N)$ with $2N$ fundamentals, is the $6d$ cousin of "vector-like matter is anomaly-free"; the second, one antisymmetric plus $N+8$ fundamentals, is the content on the tensor branch of a well-known $6d$ SCFT family. Either way the residual reducible anomaly must then factorize with suitable tensor couplings. The two solutions pass only the irreducible test; neither is a complete anomaly-consistent theory without the remaining conditions.

I.8.4 Green–Schwarz–Sagnotti–West factorization

Once the irreducible dynamical-gauge obstruction cancels, the part to be canceled by tensor inflow must *factorize* as a bilinear expression in degree-four forms,

$$I_8 \;=\; \tfrac{1}{2}\,\Omega_{ij}\, X_4^i\, X_4^j, \tag{I.8.35}$$

where $i,j = 1,\dots,n_T$ run over the tensor multiplets, Ω_{ij} is a symmetric integer matrix (the Dirac-pairing lattice metric of the self-dual string charges), and each

$$X_4^i \;=\; \tfrac{1}{4}\,a^i\, p_1 \;-\; \sum_a b_a^i\, \tfrac{1}{4}\,{\rm tr}_a F^2 \;+\; \dots \tag{I.8.36}$$

is a degree-four form carrying the Green–Schwarz coefficients a^i (the gravitational coefficient) and b_a^i (the gauge/flavor coefficients). Choose descent forms with $X_4^i =$

monomial	coefficient in I_8	origin in $\frac{1}{2}X_4^2$
p_1^2	$\frac{1}{32}$	$\frac{1}{2}(-\frac{1}{4})^2$ (square)
$(\operatorname{Tr} F_{E_8}^2)^2$	$\frac{1}{32}$	$\frac{1}{2}(-\frac{1}{4})^2$ (square)
$c_2(R)^2$	$\frac{1}{2}$	$\frac{1}{2}(1)^2$ (square)
$(\operatorname{Tr} F_{E_8}^2)\, p_1$	$\frac{1}{16}$	$(-\frac{1}{4})(-\frac{1}{4})$ (cross)
$c_2(R)\, p_1$	$-\frac{1}{4}$	$(1)(-\frac{1}{4})$ (cross)
$c_2(R)\, (\operatorname{Tr} F_{E_8}^2)$	$-\frac{1}{4}$	$(1)(-\frac{1}{4})$ (cross)
$\operatorname{tr} F_{E_8}^4$	0	irreducible: cancels by E_8 group theory

Table I.8.2: The residual anomaly eight-form I_8 of the rank-one E-string, coefficient by coefficient, with each entry's origin as a square or a cross product of the $X_4 = -\frac{1}{4}p_1 - \frac{1}{4}\operatorname{Tr} F_{E_8}^2 + c_2(R)$ coefficients. The three cross products are the nontrivial factorization checks: they are fixed once the squares fix X_4, so they are not free. The irreducible $\operatorname{tr} F^4$ vanishes because E_8 has no independent quartic Casimir. The coefficients use the stated residual normalization.

$dX_3^{i,(0)}$ and $\delta_\lambda X_3^{i,(0)} = dX_2^{i,(1)}$. With $\delta_\lambda B^i = X_2^{i,(1)}$, the classical contribution to the logarithm of the partition function is normalized as

$$\log Z_{\mathrm{GS}} \;\supset\; -\pi i \sum_{i,j} \Omega_{ij} \int_{M_6} B^i \wedge X_4^j. \tag{I.8.37}$$

For this descent choice, X_4^j is invariant. Hence $I_7^{(0)} = \frac{1}{2}\Omega_{ij} X_3^{i,(0)} \wedge X_4^j$ and $I_6^{(1)} = \frac{1}{2}\Omega_{ij} X_2^{i,(1)} \wedge X_4^j$: $dI_7^{(0)} = I_8$ and $\delta_\lambda I_7^{(0)} = dI_6^{(1)}$. Only $X_3^{i,(0)}$ varies in the latter step, so the factor $\frac{1}{2}$ remains. The variation of (I.8.37) is

$$\delta_\lambda \log Z_{\mathrm{GS}} \;=\; -\pi i \sum_{i,j} \Omega_{ij} \int_{M_6} X_2^{i,(1)} \wedge X_4^j \;=\; -\,2\pi i \int_{M_6} I_6^{(1)}, \tag{I.8.38}$$

equal and opposite to the local consistent anomaly encoded by I_8, under the stated descent and tensor-transformation rules. This local cancellation does not construct a complete chiral-tensor action or establish cancellation of global anomalies. The mechanism is the six-dimensional descendant of the $10d$ Green–Schwarz cancellation, the multi-tensor version due to Sagnotti and to Schwarz and West; the counterterm is classical, the coupling of the dynamical tensor B^i to the anomaly factor X_4^j.

The tensor fields therefore carry more than moduli: they can cancel factorized anomalies of dynamical gauge fields. A residual gauge anomaly outside the span of the available tensor couplings cannot be removed by this mechanism. Factorization constrains the tensor content and charge pairing; the E_8 background flavor terms in our example remain 't Hooft anomaly data.

Worked instance: factorizing the E-string residual

Return to the rank-one E-string, now solving (I.8.35) from the residual $I_8^{\rm res}$ of (I.8.29) rather than assuming the answer. With one tensor multiplet we take $\Omega = 1$ (a single self-dual string of unit charge, the unimodular 1×1 lattice) and seek a single $X_4 = \alpha\, p_1 + \beta\ \mathrm{Tr}\, F_{E_8}^2 + \gamma\, c_2(R)$. The three square coefficients fix the magnitudes,

$$\tfrac{1}{2}\alpha^2 = \tfrac{1}{32}, \qquad \tfrac{1}{2}\beta^2 = \tfrac{1}{32}, \qquad \tfrac{1}{2}\gamma^2 = \tfrac{1}{2}, \tag{I.8.39}$$

so $\alpha = \pm\frac{1}{4}$, $\beta = \pm\frac{1}{4}$, $\gamma = \pm 1$. Fixing the signs by the OSTY convention (the two Pontryagin/gauge coefficients negative, the R-symmetry coefficient positive) gives

$$\Omega \;=\; 1, \qquad X_4 \;=\; -\tfrac{1}{4}\, p_1 \;-\; \tfrac{1}{4}\ \mathrm{Tr}\, F_{E_8}^2 \;+\; c_2(R), \tag{I.8.40}$$

which are the Green–Schwarz coefficients $a = -1$ (from $X_4 \supset \frac{1}{4} a\, p_1$), $b_{E_8} = 1$, and $c_2(R)$-coefficient 1. So far this only used the three square coefficients; the factorization is *nontrivial* because the same α, β, γ must reproduce the three *cross* coefficients of I_8:

$$\alpha\beta \;=\; \tfrac{1}{16}, \qquad \gamma\alpha \;=\; -\tfrac{1}{4}, \qquad \gamma\beta \;=\; -\tfrac{1}{4}, \tag{I.8.41}$$

matching the $(\mathrm{Tr}\, F^2)p_1$, $c_2(R)p_1$, and $c_2(R)(\mathrm{Tr}\, F^2)$ coefficients of (I.8.29) exactly. All three hold with the solved values, so the residual genuinely factorizes. Reassembling $\frac{1}{2}\Omega X_4^2$ reproduces each stated residual coefficient. As a formal coefficient control, setting $F = p_1 = 0$ in the residual leaves $\frac{1}{2}c_2(R)^2$. This substitution tests a polynomial coefficient; it does not assert a physical background with independently realizable characteristic classes. A coefficient $2c_2(R)$ in X_4 would instead give $2c_2(R)^2$, so neither a trace rescaling nor a global sign could repair that choice. A stray p_2 or irreducible $\mathrm{tr}\, F^4$ has no home in this one-tensor square; dropping the overall $\frac{1}{2}$ or the E_8 term also changes the residual. The example shows how the tensor's bilinear term organizes an anomaly, without treating the background E_8 field as a dynamical gauge field.

I.8.5 Conformal anomalies on the tensor branch

A $6d$ conformal field theory has four Weyl-anomaly coefficients; supersymmetry constrains them. We use the Córdova–Dumitrescu–Intriligator normalization of the Weyl invariants, in which a free $(2,0)$ tensor has $c_1 = c_2 = c_3 = 1$. On a curved background the trace of the stress tensor is the six-dimensional Euler density E_6 plus three Weyl-tensor invariants I_1, I_2, I_3,

$$\langle T^{\mu}_{\ \mu} \rangle \;=\; a\, E_6 \;+\; c_1\, I_1 \;+\; c_2\, I_2 \;+\; c_3\, I_3. \tag{I.8.42}$$

There is no single $4d$-style c: six dimensions has three degree-six Weyl invariants where four dimensions had one Weyl-squared term, so (a, c_1, c_2, c_3) is the central-charge data. The free multiplets fix the normalization; their (a, c_i) contributions are comparison cases, as in Chapter 3, with two structural facts,

$$\left.\frac{c_1}{c_2}\right|_{\rm hyper} \neq \left.\frac{c_1}{c_2}\right|_{\rm tensor} \;\Rightarrow\; \text{no universal } c; \qquad a \text{ monotone (the } a\text{-theorem)}, \quad c_i \text{ not.} \tag{I.8.43}$$

The hyper and tensor ratios show that there is no single universal c. In a $\mathcal{N}=(1,0)$ superconformal theory the three c_i also obey $c_1 = \frac{1}{2}(c_2 + c_3)$ in this convention, leaving two independent c data. The $\mathcal{N}=(2,0)$ free-tensor relation used below is the decomposition

$$\mathcal{N}=(2,0)\text{ tensor } = \mathcal{N}=(1,0)\text{ tensor } \oplus \mathcal{N}=(1,0)\text{ hyper.} \tag{I.8.44}$$

These coefficients are related to the anomaly polynomial by supersymmetry. On the tensor branch, the massless multiplets and the Green–Schwarz contribution together match the origin anomaly,

$$I_8^{\mathrm{SCFT}} \;=\; I_8^{\text{massless branch}} \;+\; I_8^{\mathrm{GS}}, \tag{I.8.45}$$

where the branch spectrum can include gauge and matter multiplets as well as tensors. This is an anomaly-matching equation, not an extremization.

The extraction works because supersymmetry ties the anomalies to I_8: the R-symmetry sits in the stress-tensor multiplet, so the $c_2(R), p_1, p_2$ coefficients of the gravitational/R part

$$I_8 \;\supset\; \frac{1}{4!}\big(\alpha\, c_2(R)^2 \;+\; \beta\, c_2(R)\, p_1 \;+\; \gamma\, p_1^2 \;+\; \delta\, p_2\big) \tag{I.8.46}$$

map linearly to the conformal anomalies. The Córdova–Dumitrescu–Intriligator (CDI) relation for the Euler anomaly and the three Weyl coefficients are

$$\begin{aligned} a &= \tfrac{16}{7}(\alpha-\beta+\gamma)+\tfrac{6}{7}\delta,\\ c_1 &= 4\alpha-\tfrac{14}{3}\beta+\tfrac{16}{3}\gamma+\tfrac{8}{3}\delta,\\ c_2 &= 4\alpha-\tfrac{10}{3}\beta+\tfrac{8}{3}\gamma+\tfrac{10}{3}\delta,\\ c_3 &= 4\alpha-6\beta+8\gamma+2\delta. \end{aligned} \tag{I.8.47}$$

Each row is a fixed linear functional in the normalization of (I.8.46). The coefficient vector of $2c_1 - c_2 - c_3$ is $(0,0,0,0)$, so the supersymmetric relation holds separately in all four slots. For a free $(1,0)$ hyper the input vector is $(0,0,7/240,-1/60)$; for a free $(1,0)$ tensor it is $(1,1/2,23/240,-29/60)$. Their sum is $(1,1/2,1/8,-1/2)$, which gives $a = c_1 = c_2 = c_3 = 1$ for a free $(2,0)$ tensor. These substitutions check the normalization and the internal linear relation. The supersymmetric relation between the anomaly polynomial and conformal correlators is a physical input, not proved by these substitutions. We use its Euler row for the relative $(2,0)$ theory in the next section.

Common misconception: $6d$ has a trace-extremization principle like a-maximization

Volume 0's extremization through-line runs a-maximization ($4d$ $\mathcal{N}=1$), c-extremization ($2d$), and F-maximization ($3d$), each selecting the superconformal R by extremizing an anomaly-trace function over trial symmetries. Six dimensions hosts no such member: the (a, c_i) are fixed directly by anomaly-polynomial *factorization* and *matching* on the tensor branch, not by a variational principle, and there is no $6d$ trace whose extremum is the superconformal R. Six dimensions sits *beside* the a-max/c-ext/F-max family; the closest analog is the matching (I.8.45), an equation, not an extremum.

There is a $6d$ a-theorem: along a flow between $6d$ SCFTs the Euler coefficient decreases,

$$a_{\rm UV} \;>\; a_{\rm IR}, \tag{I.8.48}$$

the cousin of the four-dimensional a-theorem. The flow to the tensor branch is itself a $6d$ flow with $a_{\rm SCFT} > a_{\text{free branch}}$; we state (I.8.48); its proof belongs to the later anomaly analysis. The worked number is the $\mathcal{N} = (2,0)$ Euler coefficient $a \sim N^3$, computed next.

Part II. The $6d$ $\mathcal{N} = (2,0)$ world

The maximal $\mathcal{N} = (2,0)$ theory is the non-Lagrangian contrast: an ADE-classified interacting fixed point with no Lagrangian, its M5-brane origin, and the cubic conformal anomaly $a \sim N^3$.

I.8.6 The maximal $\mathcal{N} = (2,0)$ theory

The maximal supersymmetric fixed point of six dimensions is the $6d$ $\mathcal{N} = (2,0)$ theory: sixteen real supercharges (two same-chirality $6d$ Weyl supercharges), R-symmetry $USp(4)_R = Spin(5)_R$, and a single $\mathcal{N} = (2,0)$ *tensor* multiplet,

$$B^+ \;(H^+ = +\star H^+), \qquad 5 \text{ scalars in the } \mathbf{5} \text{ of } SO(5)_R, \qquad \text{fermions}, \tag{I.8.49}$$

the five scalars being transverse position moduli and B^+ the maximal-symmetry version of the $\mathcal{N} = (1,0)$ chiral tensor. The interacting theories are classified by a simply-laced (ADE) algebra $\mathfrak{g} \in \{A_n, D_n, E_6, E_7, E_8\}$, one per algebra, with tensor branch and strings

$$\mathcal{M} \;=\; (\mathbb{R}^5 \otimes \mathfrak{h})/W, \qquad \text{rank} = \text{rank}\,\mathfrak{g}, \qquad \text{tensionless strings} \;\leftrightarrow\; \text{roots}. \tag{I.8.50}$$

The A_{N-1} series is N coincident M5-branes in flat space; the D, E series need an orbifold or M-theory orientifold, discussed in later volumes. The ADE pattern is the same simply-laced structure that classified the C^2/Γ singularities and the $4d$ $\mathcal{N} = 2$ gauge symmetries. We work the A_{N-1} series; the classification proof enters later.

M5-branes and the rank. A single M5-brane carries one *free* abelian $\mathcal{N} = (2,0)$ tensor, its center-of-mass sector. Stacking N, the interacting theory is the *relative* theory after decoupling that free tensor,

$$\text{rank}\big(A_{N-1}\big) \;=\; N-1, \qquad N=1:\ \text{rank 0 (free tensor)}, \qquad A_1:\ N=2. \tag{I.8.51}$$

This is a common place to slip: a single M5 is the free tensor, not an interacting theory, and the rank-one *interacting* theory A_1 is *two* coincident M5s. Equation (I.8.51) shows why one M5 is not the interacting A_1 theory.

The absence of a Lagrangian. The chiral tensor already resists an ordinary unconstrained covariant kinetic action. On a real self-dual three-form, its naive quadratic term *vanishes*,

$$\int H \wedge \star H \;=\; \int H \wedge H \;=\; 0 \qquad (H = \star H \text{ up to signs}), \tag{I.8.52}$$

because an odd-degree form wedges with itself to zero. This on-shell identity does not prove a general no-go theorem: auxiliary-field and non-covariant descriptions exist for a free chiral tensor. No conventional local action for the interacting A_{N-1} theory is known. Its anomaly and moduli data nevertheless make it a precise quantum field theory.

> **Common misconception: no Lagrangian means it is not a field theory**
>
> The interacting $\mathcal{N} = (2,0)$ theory has no known conventional Lagrangian, and it is tempting to conclude it is not a field theory. That is wrong. It is a bona fide local quantum field theory: a stress tensor, R-symmetry $USp(4)_R$, a moduli space, BPS self-dual strings, a well-defined I_8, and conformal anomalies (a, c_i) with the sharp $a \sim N^3$ below. The absence of a Lagrangian is a statement about *presentation*, not existence. This is the $6d$ version of Chapter 2's field-theory distinction, and the partner of the $4d$ $\mathcal{N} = 2$ Argyres–Douglas fact of Chapter 5: there the obstruction was mutually non-local BPS states, here the self-dual tensor.

The tensor branch. The five scalars of each tensor multiplet are the transverse M5-brane positions in $\mathbb{R}^5$. Separating N coincident branes gives

$$\mathcal{M}_{A_{N-1}} \;=\; \big(\mathbb{R}^5\big)^{N-1}\big/S_N, \tag{I.8.53}$$

the N positions modulo the overall center of mass (leaving $N-1$ factors) and modulo the permutation S_N exchanging identical branes; its rank $N-1$ matches (I.8.51). At a generic point the theory is $N-1$ free $\mathcal{N} = (2,0)$ tensors; at the origin the interacting SCFT appears. The self-dual strings are M2-branes stretched between the M5s, with tension proportional to the separation, so they become tensionless as the branes come together, exactly the mechanism of §I.8.2.

The $\mathcal{N} = (2,0)$ anomaly polynomial. Although the $\mathcal{N} = (2,0)$ theory has no Lagrangian, its anomaly polynomial is completely known, the clearest sign that it is a genuine field theory. For the A_{N-1} theory the eight-form is (Harvey–Minasian–Moore, Intriligator)

$$I_8^{(2,0)}[A_{N-1}] \;=\; (N-1)\, I_8^{\text{free}} \;+\; \frac{N(N^2-1)}{24}\, p_2(N_R), \tag{I.8.54}$$

with I_8^{free} the free-tensor eight-form (the $N-1$ Cartan multiplets) and $p_2(N_R)$ the Pontryagin class of the $SO(5)_R$ normal bundle. Two features are load-bearing,

$$N(N^2-1) \;=\; N^3 - N \quad \text{(cubic in } N\text{)}, \qquad N(N^2-1)\big|_{N=1} \;=\; 0 \quad \text{(rank 0)}, \tag{I.8.55}$$

the cubic coefficient being the anomaly-polynomial origin of the N^3 and its vanishing at $N = 1$ the free single-M5 limit (I.8.51). The a and c_i are read off (I.8.54) by the matching of §I.8.5, and both inherit the cubic growth.

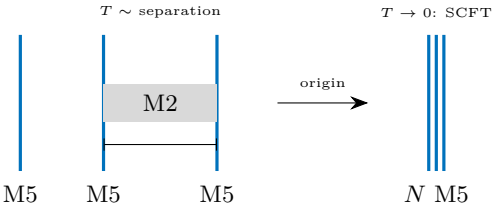


Figure I.8.2: The $\mathcal{N} = (2,0)$ A_{N-1} tensor branch as M5-brane transverse positions (here $N = 3$), schematic. Separated M5-branes give $N-1$ free tensor multiplets; an M2-brane stretched between two M5s is a self-dual string with tension proportional to the separation. As the branes coincide (right) the stretched M2 becomes a tensionless string and the interacting superconformal field theory of rank $N-1$ appears at the origin. The five transverse directions are drawn as one.

Worked instance: the $\mathcal{N} = (2,0)$ conformal anomaly $a \sim N^3$

The characteristic number of the $\mathcal{N} = (2,0)$ theory is its Euler conformal anomaly, growing as the *cube* of N, the sharpest "non-Lagrangian but field-theoretic" number in Volume 0: a theory with no Lagrangian has an exact central charge. The CDI relation (I.8.47), evaluated on the $\mathcal{N} = (2,0)$ anomaly data, gives their closed form for the Euler anomaly of the type-g theory, $a = \frac{16}{7}\, h_g^\vee d_g + r_g$ (CDI, Table 1). For $A_{N-1} = \mathfrak{su}(N)$ the dual Coxeter number, dimension, and rank are $(h^\vee, d, r) = (N,\ N^2-1,\ N-1)$, so this splits into $(N-1)$ free tensors plus the contribution supplied by anomaly matching at the interacting origin,

$$a(N) \;=\; \tfrac{16}{7}\, h^\vee d + r \;=\; (N-1)\, a_{\text{free}} \;+\; a_{\text{int}}(N), \qquad a_{\text{int}}(N) \;=\; \tfrac{16}{7}\left(N^3 - N\right), \tag{I.8.56}$$

in the normalization $a_{\text{free}} = 1$ per free $\mathcal{N} = (2,0)$ tensor (the CDI convention); with that normalization fixed, the interacting coefficient $\frac{16}{7}$ is the CDI value, not a free choice. The convention-independent content is the cubic growth $a \sim N^3$ and the tensor-branch decomposition into $(N-1)$ branch tensors plus the $\frac{16}{7}(N^3 - N)$ matching contribution. The displayed $a(N)$ belongs to the *relative* A_{N-1} theory: the overall free center-of-mass tensor of the N M5-branes has already been removed. Expanding,

$$a(N) \;=\; \tfrac{16}{7}\, N^3 \;+\; N \;-\; \tfrac{16}{7}\, N \;-\; 1 \;=\; \tfrac{16}{7}\, N^3 \;-\; \tfrac{9}{7}\, N \;-\; 1, \tag{I.8.57}$$

a cubic whose leading term comes directly from the group-theory data. The limits and the leading behaviour pin the physics,

$$a(1) = 0, \qquad a(2) = \tfrac{128}{7} - \tfrac{18}{7} - 1 = \tfrac{103}{7}, \qquad a(N) \sim \tfrac{16}{7}\, N^3, \tag{I.8.58}$$

$a(1) = 0$ the trivial relative theory beside the free single-M5 tensor, $a(2) = \frac{103}{7}$ the A_1 (two-M5) value, and $\frac{16}{7} N^3$ the large-N leading term. This beats the naive N^2 of a gauge theory,

$$\lim_{N\to\infty} \frac{a(N)}{N^3} = \tfrac{16}{7}\ \text{(finite)}, \qquad \lim_{N\to\infty} \frac{a(N)}{N^2} = \infty, \tag{I.8.59}$$

so the anomaly grows one power of N faster than the adjoint field count of an $SU(N)$ gauge theory. The exact coefficient comes from the CDI relation, while the symbolic expansion fixes its cubic degree, leading term, and values at $N = 1, 2$.

The N^3 is what makes the $\mathcal{N} = (2,0)$ theory strange. An ordinary $SU(N)$ gauge theory counts $\dim \mathfrak{su}(N) \sim N^2$ gluons, so every large-N central charge grows as N^2; the $\mathcal{N} = (2,0)$ theory has no gluons, no coupling, no Lagrangian, yet grows one power faster, both a and the c_i sharing the cubic behaviour of (I.8.54). The same N^3 appears on the gravity side as the free energy of N M5-branes in eleven-dimensional supergravity (Henningson–Skenderis), the holographic N^3 that first signalled M5-branes have no weakly-coupled field-theory description. The field-theory computation reproduces it from the anomaly polynomial alone. The construction, tensionless strings, and M-theory origin are developed later.

Compactification and the class S pointer. The $\mathcal{N} = (2,0)$ theory is the geometric origin of the $4d$ $\mathcal{N} = 2$ S-dualities of Chapter 5. Because it is conformal it has no intrinsic scale. In the four-dimensional limit below the torus Kaluza–Klein scale, reducing A_{N-1} on a two-torus T^2 of complex structure τ gives a coupling controlled by τ; the torus area sets the scale of omitted massive modes,

$$A_{N-1} \text{ on } T^2_\tau \;\longrightarrow\; 4d\ \mathcal{N} = 4\ \mathfrak{su}(N), \qquad \tau = \text{coupling}, \quad SL(2,\mathbb{Z})_{T^2} = \text{S-duality}. \tag{I.8.60}$$

Montonen–Olive duality is then the statement that two $SL(2,\mathbb{Z})$-related tori are the same torus, invisible from six dimensions, and it exists only because the parent has no coupling of its own. Compactifying instead on a punctured Riemann surface $\mathcal{C}_{g,n}$ with the partial topological twist that preserves eight supercharges gives the class S theories,

$$A_{N-1} \text{ on } \mathcal{C}_{g,n} \;\longrightarrow\; 4d\ \mathcal{N} = 2, \qquad \text{couplings} = \text{moduli}, \quad \text{S-duality} = \text{MCG}, \tag{I.8.61}$$

with the Seiberg–Witten curve of Chapter 5 a branched N-fold cover of $\mathcal{C}_{g,n}$. We develop none of the construction here; we have stated the $\mathcal{N} = (2,0)$ field-theory facts the class S program reads from.

Part III. Maximal super-Yang–Mills

The $\mathcal{N} = (1,1)$ maximal super-Yang–Mills is the Lagrangian sixteen-supercharge contrast, the non-renormalizable uplift of $4d$ $\mathcal{N} = 4$ that links to the Chapter 9 super-Yang–Mills sequence.

I.8.7 The $\mathcal{N} = (1,1)$ maximal super-Yang–Mills

The other sixteen-supercharge theory is $6d$ $\mathcal{N} = (1,1)$, and it is the opposite of $\mathcal{N} = (2,0)$ in the one way that matters here: it *is* a Lagrangian gauge theory. It has R-symmetry $Spin(4)_R = SU(2) \times SU(2)$ and is maximal $6d$ super-Yang–Mills, the uplift of $4d$ $\mathcal{N} = 4$,

$$\text{adjoint: } A_\mu,\ 4 \text{ scalars, fermions}; \qquad \text{manifest } \tfrac{1}{g^2} \operatorname{tr} F^2 \text{ action}. \tag{I.8.62}$$

But six is above the critical dimension for Yang–Mills. Since $\mathrm{tr}\, F^2$ has mass dimension four and a $6d$ Lagrangian density must have dimension six,

$$\left[\tfrac{1}{g^2}\right] \;=\; 6-4 \;=\; 2, \qquad \text{hence} \qquad [g^2] \;=\; -2, \tag{I.8.63}$$

the theory is non-renormalizable, an effective description below a cutoff, with *no* interacting $6d$ fixed point of its own; its UV completion is stringy (a little-string theory). So $\mathcal{N}=(1,1)$ is the Lagrangian sixteen-supercharge row and $\mathcal{N}=(2,0)$ the non-Lagrangian one.

One identification must be resisted. The $6d$ $\mathcal{N}=(1,1)$ SYM is *not* the $\mathcal{N}=(2,0)$ theory on a circle,

$$\begin{aligned} \mathcal{N}=(2,0) \text{ on } S^1_R \;&\longrightarrow\; 5d\ \mathcal{N}=2 \text{ maximal SYM}, \\ g_5^2 \sim R, \qquad & 1/g_5^2 \sim 1/R, \\ B^+ \to A^{(5)}_\mu, & \end{aligned} \tag{I.8.64}$$

one dimension *down*, with the Kaluza–Klein and wrapped-string modes the $5d$ instanton tower. That reduction gives $5d$, not $6d$ $\mathcal{N}=(1,1)$; the two are distinct theories.

The $\mathcal{N}=(1,1)$ SYM is the $6d$ entry of the sixteen-supercharge SYM sequence that Chapter 9 continues, $6d$ $\mathcal{N}=(1,1) \to 7d \to \cdots \to 10d$. In d dimensions the coupling dimension is

$$\left[\tfrac{1}{g^2}\right] = d-4, \qquad [g^2] = 4-d, \tag{I.8.65}$$

marginal at $d=4$ (the conformal $4d$ $\mathcal{N}=4$) and increasingly irrelevant above, as Table I.8.3 records. We state $\mathcal{N}=(1,1)$ as that contrast; the sequence is developed in Chapter 9.

The $6d \to 5d$ sequence bridge. The $\mathcal{N}=(1,0)$ theories close the dimension sequence downward too. Reducing $6d$ $\mathcal{N}=(1,0)$ on a circle gives $5d$ $\mathcal{N}=1$ (Chapter 7),

$$(\phi, B^-) \;\to\; (\phi_5, A^{(5)}_\mu), \qquad \text{self-dual string on } S^1 \;\to\; 5d \text{ instanton } (U(1)_I), \tag{I.8.66}$$

the tensor scalar and two-form descending to the $5d$ vector multiplet and the wrapped string to the $5d$ instanton tower whose charge is the Chapter 7 topological $U(1)_I$. The $6d$ $\mathcal{N}=(1,0)$ SCFTs are thus the parents of the $5d$ $\mathcal{N}=1$ theories, and the F-theory constructions that classify $6d$ SCFTs (Volume XII/XIII) read their tensor-branch, anomaly-polynomial, and Green–Schwarz data off this chapter. We state the bridge as a pointer, not a worked reduction.

Exit checklist

After this chapter the reader can

1. identify the $6d$ $\mathcal{N}=(1,0)$ multiplets (vector with no scalar, hyper, tensor with an anti-self-dual two-form), count the on-shell degrees of freedom of a self-dual two-form as the $\dim(\mathbf{1},\mathbf{3})=3$ of the little group $SO(4)$, and check that the tensor multiplet closes supersymmetry as $3+1=4$ bosonic $=4$ fermionic;

dimension	supersymmetry	$[g^2] = 4 - d$	status
$4d$	$\mathcal{N} = 4$	0	marginal: superconformal
$5d$	$\mathcal{N} = 2$	-1	irrelevant: UV completion is $6d$ $\mathcal{N} = (2,0)$ on S^1
$6d$	$\mathcal{N} = (1,1)$	-2	irrelevant: UV completion stringy (little string)
$7d$–$9d$	maximal	-3 to -5	irrelevant: no interacting fixed point
$10d$	$\mathcal{N} = 1$	-6	low-energy limit of the open superstring

Table I.8.3: The sixteen-supercharge maximal super-Yang–Mills sequence. The gauge coupling has dimension $[g^2] = 4 - d$, marginal only in four dimensions ($4d$ $\mathcal{N} = 4$). Above four dimensions maximal SYM is non-renormalizable with no interacting fixed point of its own; $6d$ $\mathcal{N} = (1,1)$ is one rung, distinct from the $\mathcal{N} = (2,0)$ theory (which on a circle gives the $5d$ row, not the $6d$ one). Chapter 9 develops the sequence; the coupling dimensions here are the elementary count $[1/g^2] = d - 4$.

2. describe the tensor branch, its dimension n_T, the tensor scalar as the inverse gauge coupling $1/g^2 \sim \langle\phi\rangle$, and the self-dual string that becomes tensionless at the origin, and distinguish it from the (absent) $6d$ Coulomb branch;

3. write the $6d$ anomaly as the eight-form I_8, assemble it from the $\hat{A}(R)\,\mathrm{ch}(F)$ building blocks and the self-dual-tensor term, and impose the irreducible $\mathrm{tr}\,F^4$ cancellation (vanishing for E_8 by $\mathrm{tr}\,F^4 = \frac{1}{100}(\mathrm{tr}\,F^2)^2$);

4. run the Green–Schwarz–Sagnotti–West factorization $I_8 = \frac{1}{2}\Omega_{ij}X_4^i X_4^j$ on the rank-one E-string, extract $\Omega = 1$ and $X_4 = -\frac{1}{4}p_1 - \frac{1}{4}\,\mathrm{Tr}\,F_{E_8}^2 + c_2(R)$, verify the three cross relations, and explain why the tensor branch is required for consistency;

5. state the four $6d$ Weyl-anomaly coefficients (a, c_1, c_2, c_3), read them off the anomaly polynomial by matching on the tensor branch, and place $6d$ beside the a-max / c-ext / F-max extremization family rather than inside it;

6. describe the $6d$ $\mathcal{N} = (2,0)$ theory (self-dual tensor multiplet, $USp(4)_R$, the ADE list), explain why it has no Lagrangian yet is a field theory, count its rank as $N - 1$ (a single M5 is the free tensor, A_1 is two M5s), and compute its conformal anomaly $a(N) = \frac{16}{7}N^3 - \frac{9}{7}N - 1 \sim N^3$;

7. contrast $6d$ $\mathcal{N} = (1,1)$ maximal super-Yang–Mills (Lagrangian, $[g^2] = -2$, no interacting fixed point) with $\mathcal{N} = (2,0)$, distinguish it from the $\mathcal{N} = (2,0)$ theory on a circle (which is $5d$ maximal SYM), and state which deep results (the anomaly-polynomial derivation, the $6d$ SCFT classification, the $\mathcal{N} = (2,0)$ construction) are cited rather than proved.

Status and sources

The chapter uses the anomaly polynomial I_{d+2}, the inflow form $\hat{A}(R)\,\mathrm{ch}(F)$ and the descent chain, with OSTY (arXiv:1408.5572) for the six-dimensional Green–Schwarz normalization. Its R-symmetries $SU(2)_R$, $USp(4)_R$ and $Spin(4)_R$ follow Chapter 1. The chiral two-form is anti-self-dual for $\mathcal{N}=(1,0)$ and self-dual for $\mathcal{N}=(2,0)$ in the chosen orientation. For the E-string, subtracting the free fields from the full source polynomial gives the residual compared with OSTY (3.41); a fixture using $2c_2(R)$ in X_4 does not establish that residual.

§I.8.1 Multiplets and the self-dual tensor. The $6d$ $\mathcal{N}=(1,0)$ vector (no scalar), hyper, and tensor multiplets (Table I.8.1); the anti-self-dual two-form $H^- = -\star H^-$ (I.8.4), the Lorentzian signature check $\star^2 = +1$ on a $6d$ three-form (I.8.6), the off-shell count $\binom{6}{2} = 15$ (I.8.7), the on-shell self-dual count $\dim(\mathbf{1},\mathbf{3}) = 3$ (half the generic 6) and the fermion $\frac{1}{2}\cdot 8 = 4$ (I.8.8)–(I.8.9), and the tensor-multiplet SUSY balance $3+1=4=4$ (I.8.10). Source anchors (Córdova et al. (2019) the $6d$ multiplet structure). Additional calculations include the self-dual dof count as an $SO(4)$ little-group rep, the SUSY $4=4$ balance, the off-shell 15, the $6d$ supercharge counts against the Chapter 1 sequence, and $\star^2=+1$.

§I.8.2 The tensor branch. The tensor branch of real dimension n_T (I.8.12), the gauge kinetic term $\phi^i \operatorname{tr} F_i^2$ and the tensor scalar as the inverse gauge coupling $1/g^2 \sim \langle\phi\rangle$ (I.8.13), and the self-dual string with tension $T \sim \langle\phi\rangle$ becoming tensionless at the origin (I.8.14). Source anchors (Seiberg (1997) the interacting $6d$ fixed points and the tensionless string; Heckman, Morrison, et al. (2014) the F-theory tensor-branch classification, the engine home). Additional calculations include the tensor-branch dimension $= n_T$ for several n_T and the zero limit of an assumed linear string-tension relation. The inverse-coupling relation is a physical input. The tensor branch is parametrized by tensor scalars, since the six-dimensional vector has no scalar.

§I.8.3 The anomaly polynomial. The $6d$ anomaly as the eight-form I_8, generated by a box diagram, the $(n+1)$-gon pattern (I.8.17) with the descent chain $8 \to 7 \to 6$; the $\hat{A}(R)\,\mathrm{ch}(F)$ building blocks (I.8.19) and the degree-eight monomial basis (I.8.22); the irreducible $\operatorname{tr} F^4$ obstruction and the E_8 identity $\operatorname{tr} F^4 = \frac{1}{100}(\operatorname{tr} F^2)^2$ (I.8.26); and the worked assembly of the rank-one E-string I_8 (I.8.29) with $[I_8]_{\operatorname{tr} F^4} = 0$ (I.8.27) obtained by full source subtraction. Source anchors (Schwarz (1996) the $6d$ $\operatorname{tr} F^4$ / factorization conditions; Ohmori, Shimizu, Tachikawa, and Yonekura (2014) the OSTY anomaly-polynomial table). The direct residual anchor is Ohmori, Shimizu, Tachikawa, and Yonekura 2014, eq. (3.41); the full rank-Q polynomial and the decoupled hyper subtraction are Ohmori, Shimizu, and Tachikawa 2014, eqs. (1.1)–(1.5), §3.3. The free tensor is in OSTY Appendix A. The new source calculation reproduces six residual coefficients and the vanishing p_2 slot; its formal $F = p_1 = 0$ control distinguishes $\frac{1}{2}c_2(R)^2$ from $2c_2(R)^2$. The E_8 quartic identity alone does not fix the E-string coefficients. The six-dimensional anomaly is encoded by the eight-form.

§I.8.4 Green–Schwarz–Sagnotti–West factorization. The factorization $I_8 = \frac{1}{2}\Omega_{ij} X_4^i X_4^j$ (I.8.35) with the degree-four form X_4^i (I.8.36), the $B \wedge X_4$ inflow coupling $\sum \Omega_{ij} \int B^i \wedge X_4^j$ (I.8.37), and the tensor branch as required for consistency; the worked E-string factorization solving $\Omega = 1$ and X_4 (I.8.40) from the square coefficients (I.8.39), with the three cross relations (I.8.41) and the reassembly against the separately subtracted OSTY polynomial. Source anchors (Green and Schwarz (1984) the $10d$ Green–Schwarz mechanism; Sagnotti (1992) the multi-tensor $6d$ factorization; Ohmori, Shimizu, Tachikawa, and Yonekura (2014) the pinned E-string coefficients). The example uses unit pairing and fixes the source coefficient vector up to an overall sign. Squaring it reproduces the residual found by subtracting the free fields from the full polynomial. The coefficient $2c_2(R)$ in X_4 fails that comparison. The example does not prove general multi-tensor factorization.

§I.8.5 Conformal anomalies on the tensor branch. The four $6d$ Weyl-anomaly coefficients (a, c_1, c_2, c_3) (I.8.42) (one Euler a, three Weyl c_i, constrained by $c_1 = \frac{1}{2}(c_2 + c_3)$ for $(1,0)$ supersymmetry in the stated normalization), related to the anomaly polynomial by supersymmetry and matched on the tensor branch (I.8.45); the $6d$ a-theorem $a_{\mathrm{UV}} > a_{\mathrm{IR}}$ stated and routed; and $6d$ placed beside the extremization family. Source anchors (Córdova et al. (2019) the $6d$ Weyl-anomaly (a, c_i) structure; Córdova et al. (2016) the $6d$ a-theorem on the tensor branch). The direct Weyl anchor is Córdova et al. 2020, eqs. (1.5), (1.12), Table 1; its Table 2 supplies the free hyper/tensor anomaly vectors used with the Euler row (1.11). The coefficient calculation yields $2c_1 - c_2 - c_3 = 0$ in all four slots and $a = c_i = 1$ for the free $(2,0)$ tensor. The physical anomaly-multiplet relation remains an input. Additional calculations include the $\mathcal{N} = (2,0)$ case of the matching, the marquee $a \sim N^3$. No $6d$ trace-extremization member is claimed (the family stays $4d$ $\mathcal{N} = 1$ / $2d$ / $3d$).

§I.8.6 The $\mathcal{N} = (2,0)$ theory. The $6d$ $\mathcal{N} = (2,0)$ tensor multiplet ($USp(4)_R$, self-dual B^+, five $SO(5)_R$ scalars) and the ADE list; the M5 rank count $\mathrm{rank}(A_{N-1}) = N - 1$ (I.8.51) (single M5 free, A_1 two M5s); the absence of a Lagrangian (the self-dual tensor obstruction); the tensor branch $(\mathbb{R}^5)^{N-1}/S_N$ (I.8.53); the marquee conformal anomaly $a(N) = \frac{16}{7}N^3 - \frac{9}{7}N - 1 \sim N^3$ (I.8.56)–(I.8.58); and the class S pointer. Source anchors (Harvey et al. (1998) the $\mathcal{N} = (2,0)$ anomaly polynomial and the N^3 scaling; Córdova et al. (2016) the $a = \frac{16}{7}h^\vee d + r$ Euler-anomaly relation and Table-1 value; Intriligator (2000) the anomaly matching and rank corrections; Henningson and Skenderis (1998) the gravity-dual N^3 cross-check). Additional calculations include $a(N)$ assembled as a polynomial, its cubic degree read off (not hard-coded), the N^3-over-N^2 growth by symbolic limits, the free ($N = 1$) and rank ($N - 1$) limits, and $A_1 =$ two M5s. The six-dimensional theory is a quantum field theory with no conventional local Lagrangian for the interacting sector.

§I.8.7 The $\mathcal{N} = (1,1)$ maximal super-Yang–Mills. $6d$ $\mathcal{N} = (1,1)$ as Lagrangian maximal SYM ($Spin(4)_R$, the uplift of $4d$ $\mathcal{N} = 4$), non-renormalizable with $[g^2] = -2$ (I.8.63) and no interacting $6d$ fixed point; the warning that it is not the $\mathcal{N} = (2,0)$ theory on a circle (that reduction is $5d$ maximal SYM); the $6d \to 5d$ sequence bridge (tensor $\to$ $5d$ vector, wrapped string $\to$ instanton tower); and the forward pointer to the Chapter 9 sixteen-supercharge SYM sequence. Source anchors (Witten (1995) the $6d$ string dynamics and the little-string / $\mathcal{N} = (2,0)$ context). Statement-level aside; the 16Q count is the count in Chapter 1 and the SYM-ladder dimension count is Chapter 9's. No CHK (a stated contrast and a routed bridge).

Exact sources for three physical pointers. The gravity-coupled condition (I.8.15) is the absence of the irreducible $\mathrm{tr}\, R^4$ term in Ferrara et al. (1998, eq. (1.1)). For (I.8.60), Assel and Schäfer-Nameki (2016, introduction and §2) gives the low-energy torus reduction, complex coupling and modular duality; Dabholkar et al. (2020, introduction p. 4) states the small-area limit at fixed complex structure. The displayed $\mathfrak{su}(N)$ is the relative sector after the free center-of-mass factor is removed. For (I.8.61), Gaiotto 2009, introduction and §4.2 gives the A-type twisted punctured-surface construction, the Seiberg–Witten covering and the coupling-space/S-duality correspondence. These are source anchors for physical pointers, not local construction proofs or new evidence tags.

Inputs and implications. The following notes state the hypotheses on dimensional reduction, multiplet assignments, and anomaly coefficients, with the scope of the corresponding calculations.

Input I.8.1. Supersymmetry spinor reality/count is imported; no spinor theorem with hypotheses is supplied.

Input I.8.2. Dimensional-reduction identification of superalgebras is imported; no controlled worldsheet derivation is given.

Input I.8.3. R-symmetry group identification is imported; no representation-theoretic proof is supplied.

Input I.8.4. Chiral-tensor sign is a local convention; classification requires separating orientation conventions from physical multiplet content.

Input I.8.5. R-representation assignments are imported; no transformation-rule derivation.

Mathematics I.8.6. Hypotheses: Hodge identity star^2=(-1)^[k(d-k)+s], d=6, k=3, s=1. Proof: Exponent 3*3+1=10 is even.

Mathematics I.8.7. Hypotheses: Six-dimensional vector space over R; no gauge quotient is taken. Proof: Choose unordered pairs of distinct indices: 6*5/2=15.

Input I.8.8. Massless little-group/chiral projection is imported; only representation dimensions are counted.

Input I.8.9. Symplectic-Majorana-Weyl state identification is imported; halving eight is only arithmetic.

Input I.8.10. Spectrum input is imported; equality 3+1=4 does not establish supersymmetry closure.

Input I.8.11. Multiplet field-content table is imported, not constructed from transformations.

Input I.8.1. Mixed table of algebra labels and spectra is imported; no uniform MATH or WS route.

Input I.8.12. Existence and independence of physical tensor scalars are imported; conditional dimension counting alone is checked.

Input I.8.13. Tensor scalar/gauge-coupling dictionary is imported; no source-level derivation in this chapter.

Input I.8.14. BPS string tension and existence of a tensionless fixed point are imported; a linear zero limit cannot establish them.

Input I.8.1. Schematic physical branch picture is imported; geometric depiction is not a verified geometry-to-QFT dictionary.

Input I.8.15. The trR4 cancellation condition is sourced, but no gravity, tensor, vector and hyper trR4 coefficients are independently assembled and matched here.

Input I.8.16. The origin/branch/GS identity is imported; no independently computed origin I8 is compared coefficient by coefficient with branch plus GS data here.

Input I.8.17. The anomaly-polynomial degree rule is imported; no six-dimensional spectrum or inflow polynomial is compared with an independent anomaly here.

Input I.8.18. The descent normalization is stated from a source; no independently computed anomalous variation is matched to its I6 descent here.

Input I.8.19. The index-density formula is imported; no fermion anomaly coefficients derived from it are matched to a separate calculation here.

Input I.8.20. The A-hat and Chern-character series are imported; no characteristic-root series proof is included.

Input I.8.21. Listing possible index-density monomials is not a coefficient match; no independent eight-form vector is compared here.

Mathematics I.8.22. Hypotheses: Assign degree 2 to F, degree 4 to p1 and c2, and degree 8 to p2. Proof: Add the degrees of the factors in each of the eight displayed monomials.

Input I.8.23. Trace decomposition is imported; representation, trace convention and independent-Casimir proof need a specific source derivation.

Input I.8.24. The quartic structure is imported; no representation coefficients are independently matched here.

Input I.8.25. The group list is imported; no per-group invariant-ring theorem/proof is supplied.

Input I.8.26. The E8 adjoint identity and 1/100 constant are imported; no weight-system proof is supplied.

Input I.8.27. The E8 trace-reduction identity is imported; no independent E8 quartic coefficient is matched to an anomaly calculation here.

Anomaly I.8.28. Six formal residual monomial slots matched; rank three independent coefficients after imposing a unit-pairing square; zero independent physical measurements. The six coefficients of the stated rank-one residual square are compared with the six coefficients obtained by subtracting the free hyper and tensor from the full Q=1 polynomial, and with OSTY (3.41). This is a finite source-polynomial match, not an inflow derivation.

Anomaly I.8.29. Six formal residual monomial slots matched; rank three independent coefficients after imposing a unit-pairing square; zero independent physical measurements. After expanding the imported X4, all six residual monomial coefficients are compared with the separately transcribed full-polynomial subtraction and OSTY (3.41); square reassembly alone would be MATH only.

Mathematics I.8.30. Hypotheses: Commuting degree-four classes p1, TrF2, c2R over Q; I8=X4^2/2. Proof: Distributivity gives squares u_i^2/2 and mixed products u_i*u_j.

Input I.8.31. Vanishing of an irreducible gauge quartic is a stated consistency condition; no independently computed B-index sum is compared here.

Input I.8.32. SU(N) quartic indices are imported; no independent weight-sum derivation is supplied.

Input I.8.33. The SU(N) quartic-index constraint uses imported group indices; no independently derived adjoint, fundamental and antisymmetric anomaly coefficients are matched here.

Mathematics I.8.34. Hypotheses: 2N-n_fund-n_antisym*(N-8)=0 with n_antisym=0 or 1. Proof: Solve linearly: n_fund=2N-n_antisym*(N-8). No existence assertion about a SCFT follows.

Anomaly I.8.2. Six formal residual monomial slots matched; rank three independent coefficients after imposing a unit-pairing square; zero independent physical measurements. Each of the table's six square/cross coefficients is compared with the corresponding independently transcribed residual-source coefficient; the table is not counted as six independent measurements.

Input I.8.35. General multi-tensor factorization is a sourced premise; the rank-one E-string comparison does not match coefficients for this general assertion.

Input I.8.36. The displayed general X4 sign convention is imported from OSTY; the E-string coefficient comparison tests a special X4 and cannot establish a coefficient match for arbitrary tensor/gauge data.

Input I.8.37. The local $-\pi i\,\Omega\int BX_4$ coefficient follows from the stated descent choice and $\delta B = X_2$; its physical chiral-tensor coupling is imported, and no complete action or global anomaly claim follows.

Input I.8.38. The half-factor in $I_6^{(1)}$ gives formal local cancellation; no independent one-loop variation vector is compared with a fermion anomaly, and global anomaly cancellation is not established.

Mathematics I.8.39. Hypotheses: Real coefficients; half their squares are (1/32,1/32,1/2). Proof: Multiply by two and take nonnegative square roots; cross terms fix relative signs.

Anomaly I.8.40. Six formal residual monomial slots matched; rank three independent coefficients after imposing a unit-pairing square; zero independent physical measurements. The solved X4 branch reproduces the six residual-source slots, including the two R-mixed coefficients that reject the old 2c2(R) input; a global sign is invisible to this polynomial match.

Mathematics I.8.41. Hypotheses: Unit pairing, commuting formal classes, and the stated X4 sign branch. Proof: Multiply alpha*beta, gamma*alpha, gamma*beta. Simultaneous sign reversal leaves each product invariant.

Input I.8.42. The Weyl-coefficient count, free-tensor normalization and supersymmetric relation are sourced; conditional CDI row arithmetic is not a separate physical anomaly match.

Input I.8.43. Free-multiplet c-ratios and Euler monotonicity are sourced statements; no independent flow or free-field anomaly coefficient comparison is performed for this atom.

Input I.8.44. Physical multiplet decomposition is imported; adding anomaly vectors does not prove representation equivalence.

Input I.8.45. The tensor-branch matching rule is imported; no origin I8 is independently computed and compared with the massless and GS eight-forms here.

Input I.8.46. The four-monomial CDI normalization is imported; no independent alpha, beta, gamma, delta anomaly vector is compared here.

Input I.8.47. The CDI physical anomaly-multiplet relation is imported; four-slot row cancellation and free-(2,0) normalization are conditional algebra, with no independent SCFT anomaly-vector comparison.

Input I.8.48. Euler monotonicity is an imported flow theorem; no independently computed UV and IR a coefficients are matched for a specified flow here.

Input I.8.49. USp(4) multiplet identification is imported; no transformation-rule proof.

Input I.8.50. ADE classification and tensor-branch realization are imported; no construction or classification proof.

Input I.8.51. M5/relative-theory identification is imported; N-1 arithmetic cannot establish that dictionary.

Mathematics I.8.52. Hypotheses: Ordinary real differential three-form on an oriented Lorentzian six-manifold; star H=+/-H. Proof: Graded commutativity gives H wedge H=(-1)^9 H wedge H, hence zero over R. Substitute star H=+/-H; integration is zero when defined. No action no-go is implied.

Input I.8.53. Separated-brane/moduli-space dictionary is imported; no covering-geometry argument supplied.

Input I.8.2. M2/self-dual-string dictionary is imported; the figure is schematic and no brane calculation is performed.

Input I.8.54. The relative A-series inflow polynomial is imported; the chapter does not independently compute and match its p2(N_R) coefficient.

Mathematics I.8.55. Hypotheses: Formal variable N; use the imported inflow coefficient as a polynomial. Proof: Expand to N^3-N and substitute N=1.

Input I.8.56. The relative A-series Euler formula combines imported CDI and group data; collecting polynomial terms is conditional arithmetic, not an independent anomaly match.

Mathematics I.8.57. Hypotheses: a=(N-1)+(16/7)(N^3-N); the physical CDI/M5 input is imported. Proof: Distribute and collect N terms: 1-16/7=-9/7.

Input I.8.58. The N=1, N=2 and asymptotic values follow from the imported Euler formula; no independently obtained anomaly coefficients are matched at those ranks.

Mathematics I.8.59. Hypotheses: N positive and tending to infinity; a=(16/7)N^3-(9/7)N-1. Proof: Divide termwise: lower inverse powers vanish, while a/N^2 has positive linear leading term.

Input I.8.60. Torus reduction transfers a physical theory; no typed duality-chain evidence is supplied.

Input I.8.61. Twisted class-S construction and mapping-class dualities are imported; no construction or chain proof.

Input I.8.62. Maximal SYM identification and field content are imported; no supersymmetry transformations supplied.

Mathematics I.8.63. Hypotheses: Action integral d^6x tr(F^2)/g^2; [x]=-1 and [F]=2. Proof: Zero action dimension requires -6+4-[g^2]=0.

Input I.8.64. Circle reduction to 5d maximal SYM is imported; no KK/instanton derivation supplied.

Mathematics I.8.65. Hypotheses: Action integral d^dx tr(F^2)/g^2; [x]=-1 and [F]=2. Proof: Zero action dimension requires -d+4-[g^2]=0.

Input I.8.3. The table mixes conditional dimensions with physical supersymmetry/content assignments; only [g^2]=4-d is derived above.

Input I.8.66. Tensor/vector reduction and wrapped-string/instanton identification are imported; no reduction calculation.

Input I.8.2. The absence of vector scalars and distinction of physical branches are imported multiplet facts.

Input I.8.2. Linear tensor-scalar/gauge-coupling dictionary is imported; no construction of the coupling matrix.

Input I.8.3. The background-versus-dynamical anomaly distinction is stated; no E8 flavor coefficient is matched to an independent calculation for this prose atom.

Anomaly I.8.3. Six formal residual monomial slots matched; rank three independent coefficients after imposing a unit-pairing square; zero independent physical measurements. Full Q=1 polynomial minus its decoupled hyper and the free tensor agrees in six residual slots with OSTY (3.41); c2(L) and p2 cancel in the same subtraction, with no extra deletion.

Input I.8.5. No trace extremization is used; the broader absence assertion is imported, not established by the anomaly computation.

Input I.8.6. Claims about known action formulations are imported; the odd-form identity is not a no-go proof.

Input I.8.7. Little-string UV completion is imported; negative coupling dimension establishes only a perturbative effective-theory limitation.

The free-field coefficient comparison includes a separate conditional calculation [MATH]: for the four displayed CDI rows and free-field vectors, componentwise subtraction gives $(2c_1 - c_2 - c_3)_{\alpha,\beta,\gamma,\delta} = (0,0,0,0)$, and rational dot products give $(a, c_1, c_2, c_3) = (1,1,1,1)$. Its physical component remains a sourced supersymmetric relation. **Routed (stated, not proved).** The anomaly-polynomial building-block derivation, the general Green–Schwarz-factorization theorem, the full $6d$ anomaly-inflow argument, and the $6d$ a-theorem; the construction of the interacting $\mathcal{N} = (2,0)$ theory, the tensionless-string dynamics, the M5-brane / M-theory origin, and the class S compactification of $\mathcal{N} = (2,0)$ on a punctured surface; and the F-theory tensor-branch classification of $6d$ $\mathcal{N} = (1,0)$ SCFTs (the engine home, Volume XII/XIII, named not stubbed). All are stated and cited here; their proofs and constructions live in Part III and the engine volumes.

Further reading

The $\mathcal{N} = (2,0)$ theory was identified in Witten (1995), with tensionless strings from Strominger (1996). Its anomaly polynomial and N^3 scaling are in Harvey et al. (1998) and Yi (2001), the general six-dimensional case in Ohmori, Shimizu, Tachikawa, and Yonekura (2014), and the E-string in Ohmori, Shimizu, and Tachikawa (2014). The Euler anomaly and the six-dimensional a-theorem are in Córdova et al. (2016). The classification of $6d$ $\mathcal{N} = (1,0)$ theories and conformal matter is in Heckman, Morrison, et al. (2014), Del Zotto et al. (2015), and Bhardwaj (2015), reviewed in Heckman and Rudelius (2019).

More recent refinements of the $6d$ classification appear in Bhardwaj (2020); modern treatments of higher-group symmetry and anomaly matching in six dimensions are Córdova et al. (2021) and Apruzzi et al. (2022), while the bordism-based status of global gauge anomalies is clarified in Davighi and Lohitsiri (2021).

References

Apruzzi, F. et al. (2022). "2-Group symmetries and their classification in 6d". In: *SciPost Phys.* 12.3, p. 098. DOI: 10.21468/SciPostPhys.12.3.098. arXiv: 2110.14647 [hep-th].

Assel, B. and S. Schäfer-Nameki (2016). "Six-dimensional origin of $\mathcal{N} = 4$ SYM with duality defects". In: *JHEP* 12, p. 058. DOI: 10.1007/JHEP12(2016)058. arXiv: 1610.03663 [hep-th].

Bhardwaj, L. (2015). "Classification of 6d $\mathcal{N} = (1,0)$ gauge theories". In: *JHEP* 11, p. 002. DOI: 10.1007/JHEP11(2015)002. arXiv: 1502.06594 [hep-th].

Bhardwaj, L. (2020). "Revisiting the classifications of 6d SCFTs and LSTs". In: *JHEP* 03, p. 171. DOI: 10.1007/JHEP03(2020)171. arXiv: 1903.10503 [hep-th].

Córdova, C., T. T. Dumitrescu, and K. Intriligator (2016). "Anomalies, renormalization group flows, and the a-theorem in six-dimensional (1,0) theories". In: *JHEP* 10, p. 080. DOI: 10.1007/JHEP10(2016)080. arXiv: 1506.03807 [hep-th].

Córdova, C., T. T. Dumitrescu, and K. Intriligator (2019). "Multiplets of Superconformal Symmetry in Diverse Dimensions". In: *JHEP* 03, p. 163. DOI: 10.1007/JHEP03(2019)163. arXiv: 1612.00809 [hep-th].

Córdova, C., T. T. Dumitrescu, and K. Intriligator (2020). "$\mathcal{N} = (1, 0)$ Anomaly Multiplet Relations in Six Dimensions". In: *JHEP* 07, p. 065. DOI: 10.1007/JHEP07(2020)065. arXiv: 1912.13475 [hep-th].

Córdova, C., T. T. Dumitrescu, and K. Intriligator (2021). "2-Group Global Symmetries and Anomalies in Six-Dimensional Quantum Field Theories". In: *JHEP* 04, p. 252. DOI: 10.1007/JHEP04(2021)252. arXiv: 2009.00138 [hep-th].

Dabholkar, A., P. Putrov, and E. Witten (2020). "Duality and Mock Modularity". In: *SciPost Phys.* 9.5, p. 072. DOI: 10.21468/SciPostPhys.9.5.072. arXiv: 2004.14387 [hep-th].

Davighi, J. and N. Lohitsiri (2021). "Omega vs. pi, and 6d anomaly cancellation". In: *JHEP* 05, p. 267. DOI: 10.1007/JHEP05(2021)267. arXiv: 2012.11693 [hep-th].

Del Zotto, M. et al. (2015). "6d Conformal Matter". In: *JHEP* 02, p. 054. DOI: 10.1007/JHEP02(2015)054. arXiv: 1407.6359 [hep-th].

Ferrara, S., F. Riccioni, and A. Sagnotti (1998). "Tensor and vector multiplets in six-dimensional supergravity". In: *Nucl. Phys. B* 519, pp. 115–140. DOI: 10.1016/S0550-3213(97)00837-7. arXiv: hep-th/9711059.

Gaiotto, D. (2009). *N=2 dualities*. arXiv: 0904.2715 [hep-th].

Green, M. B. and J. H. Schwarz (1984). "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory". In: *Phys. Lett. B* 149, pp. 117–122. DOI: 10.1016/0370-2693(84)91565-X.

Harvey, J. A., R. Minasian, and G. W. Moore (1998). "NonAbelian tensor multiplet anomalies". In: *JHEP* 09, p. 004. DOI: 10.1088/1126-6708/1998/09/004. arXiv: hep-th/9808060.

Heckman, J. J., D. R. Morrison, and C. Vafa (2014). "On the Classification of 6D SCFTs and Generalized ADE Orbifolds". In: *JHEP* 05. [Erratum: JHEP 06, 017 (2015)], p. 028. DOI: 10.1007/JHEP05(2014)028. arXiv: 1312.5746 [hep-th].

Heckman, J. J. and T. Rudelius (2019). "Top Down Approach to 6D SCFTs". In: *J. Phys. A* 52.9, p. 093001. DOI: 10.1088/1751-8121/aafc81. arXiv: 1805.06467 [hep-th].

Henningson, M. and K. Skenderis (1998). "The Holographic Weyl anomaly". In: *JHEP* 07, p. 023. DOI: 10.1088/1126-6708/1998/07/023. arXiv: hep-th/9806087.

Intriligator, K. (2000). "Anomaly matching and a Hopf-Wess-Zumino term in 6d, N=(2,0) field theories". In: *Nucl. Phys. B* 581, pp. 257–273. DOI: 10.1016/S0550-3213(00)00148-6. arXiv: hep-th/0001205.

Ohmori, K., H. Shimizu, and Y. Tachikawa (2014). "Anomaly polynomial of E-string theories". In: *JHEP* 08, p. 002. DOI: 10.1007/JHEP08(2014)002. arXiv: 1404.3887 [hep-th].

Ohmori, K., H. Shimizu, Y. Tachikawa, and K. Yonekura (2014). "Anomaly polynomial of general 6d SCFTs". In: *PTEP* 2014.10, 103B07. DOI: 10.1093/ptep/ptu140. arXiv: 1408.5572 [hep-th].

Sagnotti, A. (1992). “A Note on the Green-Schwarz mechanism in open string theories”. In: *Phys. Lett. B* 294, pp. 196–203. DOI: 10.1016/0370-2693(92)90682-T. arXiv: hep-th/9210127.

Schwarz, J. H. (1996). “Anomaly-free supersymmetric models in six dimensions”. In: *Phys. Lett. B* 371, pp. 223–230. DOI: 10.1016/0370-2693(95)01610-4. arXiv: hep-th/9512053.

Seiberg, N. (1997). “Nontrivial fixed points of the renormalization group in six dimensions”. In: *Phys. Lett. B* 390, pp. 169–171. DOI: 10.1016/S0370-2693(96)01424-4. arXiv: hep-th/9609161.

Strominger, A. (1996). “Open p-branes”. In: *Phys. Lett. B* 383. Ed. by M. J. Duff, pp. 44–47. DOI: 10.1016/0370-2693(96)00712-5. arXiv: hep-th/9512059.

Witten, E. (1995). “Some comments on string dynamics”. In: arXiv: hep-th/9507121.

Yi, P. (2001). “Anomaly of (2,0) theories”. In: *Phys. Rev. D* 64, p. 106006. DOI: 10.1103/PhysRevD.64.106006. arXiv: hep-th/0106165.

Chapter I.9

$7d$ to $10d$ supersymmetric field theories

A gauge field in ten spacetime dimensions has eight physical polarizations. Its Majorana–Weyl superpartner has eight on-shell components as well, giving the $10d$ $\mathcal{N} = 1$ classical super Yang–Mills multiplet. What becomes of this multiplet when one, two, or three spatial directions are reduced? Each lost component of the gauge field becomes an adjoint scalar. The resulting $9d$, $8d$, and $7d$ theories still carry sixteen supercharges, while their scalar potentials, R-symmetries, and vacuum spaces change with the number of reduced directions.

We follow the fields through each reduction, including the spinor content, coupling, and terms of the action. The same calculation gives the compact relations $\#\phi = 10 - d$, $[g_d^2] = 4 - d$, and $\dim_{\mathbb{R}} \mathcal{M}_{\text{Coulomb}} = (10 - d) \operatorname{rank} G$. They distinguish the four theories without introducing four unrelated multiplets. The $8d$ theory supplies the local vector multiplet of a 7-brane, while the $7d$ theory supplies that of an ADE singularity in M-theory. These connections identify the field theories used by those constructions; the constructions themselves enter later volumes.

Take-home message of this chapter

Reducing $10d$ $\mathcal{N} = 1$ super Yang–Mills on T^{10-d} gives one sixteen-supercharge vector multiplet in each dimension $7 \leq d \leq 10$. Its $10 - d$ adjoint scalars arise from internal gauge-field components, and their commuting expectation values form, locally, the real space $\mathbb{R}^{(10-d)r}/W_G$. In the local reduced theory, transverse rotations act on those scalars; their spin cover acts on the supercharges, giving $U(1)_R$ in $8d$ and $SU(2)_R$ in $7d$. In every row $[g_d^2] = 4 - d < 0$.

Chapter 1 supplies the supercharge and Lorentzian spinor conventions used below. Its high-dimensional rows all have sixteen real supercharges. Here we derive their field content from the ten-dimensional parent and explain why the R-symmetry and vacuum spaces change as the dimension decreases.

I.9.1 $10d$ $\mathcal{N}=1$ super Yang–Mills: the parent

The classical Lagrangian parent of this reduction is $10d$ $\mathcal{N}=1$ super Yang–Mills. Its entire field content is one multiplet, which we list explicitly rather than describe in passing:

Pure nonabelian $10d$ SYM is anomalous as a standalone quantum theory. Here its classical action supplies the local reduction formulas; the heterotic and type I gauge sectors belong to anomaly-cancelled string backgrounds with additional string and gravity degrees of freedom. The lower-dimensional reductions have their own quantum status.

- *The* $10d$ $\mathcal{N}=1$ *vector multiplet* (sixteen supercharges; the unique multiplet without gravity in $10d$). Constraint: the gaugino is Majorana–Weyl with the fixed chirality $\Gamma_{11}\lambda = +\lambda$ (the heterotic and type I gaugino chirality of Chapter 1). Field content: a gauge field A_M, $M = 0, 1, \ldots, 9$, in the adjoint of the gauge group G, and the adjoint gaugino λ; no scalars, no other fermions. Role: this one object, in $10d$ or reduced, is every theory in this chapter.

The Lagrangian is the minimal gauge-covariant action for these two fields,

$$\mathcal{L} \;=\; -\frac{1}{4g^2}\,\mathrm{Tr}\,F_{MN}F^{MN} \;+\; \frac{i}{2g^2}\,\mathrm{Tr}\,\bar{\lambda}\,\Gamma^M D_M \lambda, \tag{I.9.1}$$

with $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$ the field strength and D_M the adjoint gauge-covariant derivative, in the mostly-plus signature of Chapter 1. Throughout this chapter Tr is a positive invariant trace normalized by $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ for Hermitian generators, so for $SU(N)$ $\mathrm{Tr} = 2\mathrm{Tr}_f$ when $\mathrm{Tr}_f(T^a T^b) = \delta^{ab}/2$. Thus the gauge term is $-F^a_{MN}F^{aMN}/(4g^2)$; this same Tr is used in the reduced action and potential. The two terms are the unique gauge-invariant kinetic terms, and supersymmetry fixes their relative coefficient. We do not vary the action here; Chapter 2 fixed the multiplet grammar. What we do check is the one algebraic consistency condition any supersymmetric theory must satisfy: on-shell bosonic and fermionic degrees of freedom must match.

The on-shell degree-of-freedom match. A massless gauge field in D spacetime dimensions has $D-2$ physical transverse polarizations, after gauge-fixing and the equations of motion remove the timelike and longitudinal modes. At $D = 10$,

$$\#\,A_M\big|_{\text{on-shell}} \;=\; D-2 \;=\; 8. \tag{I.9.2}$$

The gaugino count is a four-step chain, and it is worth displaying every step once. A Dirac spinor of $\mathrm{Spin}(1,9)$ has $2^{\lfloor 10/2 \rfloor} = 32$ complex components, that is 64 real ones. The Majorana condition is a reality condition and halves the real count. The Weyl condition $\Gamma_{11}\lambda = +\lambda$ halves it again (the two conditions are compatible precisely in $D = 2 \bmod 8$, so at $D = 10$ in this chapter's range):

$$64 \;\xrightarrow{\text{Majorana}}\; 32 \;\xrightarrow{\text{Weyl}}\; 16 \qquad \text{real off-shell components.} \tag{I.9.3}$$

On-shell, the Dirac equation halves the count once more:

$$\#\,\lambda\big|_{\text{on-shell}} \;=\; \frac{16}{2} \;=\; 8. \tag{I.9.4}$$

The bosonic and fermionic counts agree,

$$8 \;=\; 8, \tag{I.9.5}$$

the smallest sixteen-supercharge consistency check and the anchor for everything downstream. Neither 8 is put in by hand: the polarization count is $D-2$ and the spinor count is the chain (I.9.3) halved, both evaluated at $D=10$. The same two counting functions carry the match down the reduction sequence, where the gauge field loses polarizations and scalars appear to replace them.

Why there is no $11d$ super Yang–Mills. The match is genuinely a statement about D, and the neighboring dimension shows it. One must be careful with the formulas here: 11 is odd, so there is no Weyl condition and the Majorana–Weyl chain (I.9.3) does not apply. The correct $11d$ counting starts from the same $2^{\lfloor 11/2\rfloor}=32$ complex Dirac components; the minimal $11d$ spinor is Majorana, with 32 real components, and the Dirac equation halves it on-shell:

$$64 \;\xrightarrow{\text{Majorana}}\; 32 \;\xrightarrow{\text{on-shell}}\; 16. \tag{I.9.6}$$

A putative $11d$ SYM multiplet would have to match these 16 fermionic states against a gauge field alone, which supplies only

$$D-2 \;=\; 9 \;\neq\; 16, \tag{I.9.7}$$

and a pure gauge multiplet has no scalars available to make up the difference. So the minimal $11d$ multiplet is not a gauge multiplet at all: it is the *supergravity* multiplet, whose counting does close,

$$\underbrace{44}_{g_{MN}} + \underbrace{84}_{C_{MNP}} \;=\; 128 \;=\; \underbrace{128}_{\psi_M}, \tag{I.9.8}$$

the graviton ($\frac{1}{2}(D-2)(D-1)-1=44$), the three-form ($\binom{D-2}{3}=84$), and the Majorana gravitino ($(D-3)\times 16=128$) at $D=11$. Eleven dimensions is the home of supergravity, forced by the $32Q$ ceiling of Chapter 1, and $10d$ is the ceiling for a gauge theory. The degree-of-freedom arithmetic and the supercharge ceiling agree: $10d$ $\mathcal{N}=1$ SYM is the top.

The coupling has negative mass dimension. The Yang–Mills coupling g^2 carries a mass dimension. With the gauge field at its geometric dimension $[A]=1$, so $[F]=2$ and $[\mathrm{Tr}\,F^2]=4$, the requirement that $\int d^dx\,\mathcal{L}$ be dimensionless fixes the prefactor:

$$\left[\frac{1}{g^2}\right]+4 \;=\; d \qquad\Longrightarrow\qquad \left[\frac{1}{g^2}\right] \;=\; d-4, \tag{I.9.9}$$

that is,

$$[g^2] \;=\; 4-d. \tag{I.9.10}$$

At $d = 10$ this is

$$[g^2] \;=\; 4 - 10 \;=\; -6 \;<\; 0. \tag{I.9.11}$$

A negative coupling dimension makes the Yang–Mills interaction perturbatively non-renormalizable. Dimensional analysis identifies the scale at which its dimensionless effective coupling becomes order one,

$$\Lambda \;\sim\; (g^2)^{1/(4-d)} \;=\; (g_{10}^2)^{-1/6} \quad \text{at } d = 10, \tag{I.9.12}$$

where the perturbative SYM description ceases to be weakly coupled. For the heterotic and type I gauge sectors, string degrees of freedom supply the corresponding ultraviolet description. This scale estimate by itself does not establish the detailed completion of an arbitrary choice of gauge group.

No Coulomb branch, no continuous R-symmetry. Two absences at the parent level shape the rest of the chapter. First, there is no Coulomb branch. A Coulomb branch is parametrized by the vacuum expectation values of adjoint scalars in the vector multiplet, but the $10d$ vector multiplet has no scalar: there is nothing to give a vacuum expectation value, so the moduli space is a point. Second, there is no continuous R-symmetry. The R-symmetry of a reduced theory will turn out to be the rotation group of the reduced directions, but at $d = 10$ there are no reduced directions: the transverse rotation group is $SO(10 - 10) = SO(0)$, the trivial group. Both absences are undone by reduction, and understanding why is the content of the next three sections. In $10d$ the theory is rigid: one multiplet, one coupling, no moduli, no continuous R-symmetry.

Worked instance: the $10d$ degree-of-freedom match

The $10d$ $\mathcal{N} = 1$ vector multiplet carries, on-shell, a gauge field with $D - 2 = 8$ transverse polarizations and a Majorana–Weyl gaugino with $64 \to 32 \to 16 \to 8$ real components after the Majorana, Weyl, and on-shell halvings, so bosons match fermions, $8 = 8$. The match fails at $D = 11$: the minimal spinor there is Majorana with 32 real components, giving 16 on-shell fermionic states against only 9 gauge polarizations and no scalars, so the minimal $11d$ multiplet is the supergravity multiplet (I.9.8), not a SYM multiplet. The identity $(d - 2) + (10 - d) = 8$ carries the bosonic total unchanged down the reduction sequence.

I.9.2 The reduction sequence $10d \to 7d$

At each reduction step, one component of the gauge field along the new circle becomes a scalar in the lower-dimensional theory. We derive this from the zero modes and the action, then apply it successively to the $9d$, $8d$, and $7d$ theories. The first step has no scalar commutator; the next two introduce the potential that controls the vacuum space.

The mechanism: zero modes on a torus. Reduction means the simplest possible operation. Place the $10d$ theory on the product spacetime

$$\mathbb{R}^{1,d-1} \times T^{10-d}, \qquad x^m \;\simeq\; x^m + 2\pi R_m, \quad m = d, \ldots, 9, \tag{I.9.13}$$

a d-dimensional Minkowski factor times a torus of the remaining $10-d$ directions. Every field can be Fourier-expanded along each circle,

$$\Phi(x^\mu, x^m) \;=\; \sum_{n\in\mathbb{Z}} \Phi^{(n)}(x^\mu)\, e^{inx^m/R_m}, \tag{I.9.14}$$

and a mode with Fourier number n carries a Kaluza–Klein mass set by the circle size,

$$m_{\mathrm{KK}} \;=\; \frac{|n|}{R_m}. \tag{I.9.15}$$

Reduction keeps only the constant modes $n=0$, the fields that do not vary along the torus, which is the same as imposing $\partial_m = 0$ on everything. The supersymmetry parameter ϵ is also a constant spinor, untouched by the truncation, so all 16 real supercharges survive: the reduced theory carries the same 16 supercharges as the parent at every step (the Chapter 1 reduction map, recalled, not re-derived).

The gauge-field split. The $10d$ gauge field A_M has its vector index split by the reduction into a d-dimensional part and a transverse part,

$$A_M \;\longrightarrow\; \big(\,A_\mu,\ \phi^m\,\big), \qquad \mu = 0,\dots,d-1, \quad m = d,\dots,9. \tag{I.9.16}$$

The d components A_μ remain a d-dimensional gauge field. The $10-d$ components $\phi^m \equiv A_m$ no longer carry a d-dimensional vector index; from the d-dimensional point of view each is a scalar. The gauge transformations that survive the truncation are the x^m-independent ones, and under them the two pieces transform differently,

$$A_\mu \;\to\; U A_\mu U^{-1} + i\,U\partial_\mu U^{-1}, \qquad \phi^m \;\to\; U\phi^m U^{-1}\,: \tag{I.9.17}$$

A_μ as a gauge field, each ϕ^m as an adjoint scalar. So reduction produces

$$\#\,\text{adjoint scalars} \;=\; 10-d. \tag{I.9.18}$$

These ϕ^m are the seed of everything the parent lacked: the R-symmetry will rotate them, and their vacuum expectation values will parametrize the Coulomb branch.

The field strength and the action, in general. With $\partial_m = 0$, the mixed and transverse components of F_{MN} collapse to covariant derivatives and commutators of the scalars:

$$F_{\mu m} \;=\; \partial_\mu\phi^m - i[A_\mu,\phi^m] \;=\; D_\mu\phi^m, \qquad F_{mn} \;=\; -\,i\,[\phi^m,\phi^n]. \tag{I.9.19}$$

Substituting into the gauge kinetic term and separating the index sums ($F_{MN}F^{MN} = F_{\mu\nu}F^{\mu\nu} + 2F_{\mu m}F^{\mu m} + F_{mn}F^{mn}$) gives the master split, with repeated m,n summed:

$$-\frac{1}{4g^2}\operatorname{Tr} F_{MN}F^{MN} = -\frac{1}{4g^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\operatorname{Tr} D_\mu\phi^m D^\mu\phi^m + \frac{1}{4g^2}\operatorname{Tr}[\phi^m,\phi^n][\phi^m,\phi^n]. \tag{I.9.20}$$

Each term has a clear origin: the first is the d-dimensional gauge kinetic term; the second is the scalar kinetic term, born from the mixed components $F_{\mu m} = D_\mu\phi^m$ (the

factor 2 from the two index orders becomes the standard $\frac{1}{2}$); the third is born from the purely transverse components $F_{mn} = -i[\phi^m, \phi^n]$ and is the scalar potential. For Hermitian scalars the commutator is anti-Hermitian, so $\mathrm{Tr}\,[\phi^m, \phi^n]^2 \leq 0$ and the last term is minus a non-negative potential; §I.9.4 takes it up. The reduction also preserves the on-shell degree-of-freedom count at every step: the gauge field contributes $d-2$ transverse polarizations and the scalars contribute $10-d$, so the bosonic total is

$$(d-2) \;+\; (10-d) \;=\; 8, \tag{I.9.21}$$

independent of d, matched at every row by the 8 on-shell components of the reduced gaugino. This is the sense in which the whole reduction sequence is one multiplet: the same $8+8$ on-shell degrees of freedom, packaged differently at each dimension. We now watch the packaging change, one circle at a time.

Worked instance: the $10d \to 9d$ step, in full

Compactify the single direction x^9 on a circle of radius R_9 and keep the zero modes, $\partial_9 = 0$.

(a) The index split. The ten components of A_M split as

$$A_M \;\longrightarrow\; \big(A_\mu,\ \phi^9\big), \qquad \mu = 0, \ldots, 8, \quad \phi^9 \equiv A_9, \tag{I.9.22}$$

a $9d$ gauge field plus one adjoint scalar: the scalar count is $10-9=1$. It is instructive to watch all $\binom{10}{2} = 45$ independent components of the antisymmetric F_{MN} land somewhere:

$$45 \;=\; \underbrace{36}_{F_{\mu\nu}} \;+\; \underbrace{9}_{F_{\mu 9}} \;+\; \underbrace{0}_{F_{mn}}, \tag{I.9.23}$$

$\binom{9}{2} = 36$ gauge components, $9 \times 1 = 9$ mixed components, and no purely transverse pair, because a single reduced direction admits no antisymmetric pair. Nothing is lost and nothing is added; the components are relabeled.

(b) The mixed field strength is the scalar's covariant derivative. With $\partial_9 = 0$,

$$F_{\mu 9} \;=\; \partial_\mu A_9 - \partial_9 A_\mu - i[A_\mu, A_9] \;=\; \partial_\mu \phi^9 - i[A_\mu, \phi^9] \;=\; D_\mu \phi^9, \tag{I.9.24}$$

the middle term dying by the zero-mode condition. This is the general statement (I.9.19) caught in the act.

(c) The action splits. Substituting into the kinetic term,

$$\mathrm{Tr}\, F_{MN} F^{MN} \;=\; \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \;+\; 2\,\mathrm{Tr}\, D_\mu \phi^9 D^\mu \phi^9, \tag{I.9.25}$$

with no potential term: one scalar has no partner to fail to commute with ($\binom{1}{2} = 0$ commutator pairs, the 0 of (I.9.23)). The x^9 integral in the action contributes the circle volume, $\int d^{10}x = 2\pi R_9 \int d^9 x$, so

$$S \;=\; -\frac{2\pi R_9}{4 g_{10}^2} \int d^9 x\; \mathrm{Tr}\Big(F_{\mu\nu} F^{\mu\nu} + 2\, D_\mu \phi^9 D^\mu \phi^9 \Big) \;+\; \ldots \tag{I.9.26}$$

(d) The coupling and its dimension. Reading off the prefactor of the $9d$ gauge kinetic term defines the $9d$ coupling, and the circle length shifts the dimension by exactly one:

$$\frac{1}{g_9^2} \;=\; \frac{2\pi R_9}{g_{10}^2} \qquad\Longrightarrow\qquad [g_9^2] \;=\; [g_{10}^2] + 1 \;=\; -5 \;=\; 4 - 9, \tag{I.9.27}$$

consistent with the general dimension count (I.9.10) run directly at $d = 9$. The coupling is still irrelevant; descending has made it less negative by one unit, not positive.

(e) The gaugino reduces and a Yukawa coupling appears. The same zero-mode condition collapses the gaugino's kinetic term. Since $D_9\lambda = -i[\phi^9, \lambda]$,

$$\frac{i}{2g_{10}^2}\,\mathrm{Tr}\,\bar\lambda\,\Gamma^M D_M\lambda \;\longrightarrow\; \frac{i}{2g_9^2}\,\mathrm{Tr}\,\bar\lambda\,\Gamma^\mu D_\mu\lambda \;+\; \frac{1}{2g_9^2}\,\mathrm{Tr}\,\bar\lambda\,\Gamma^9[\phi^9, \lambda], \tag{I.9.28}$$

a $9d$ kinetic term plus the scalar–gaugino Yukawa coupling that supersymmetry demands. As a representation of the $9d$ Lorentz group, the 16 real components of λ simply restrict: the transverse group Spin(1) is trivial, so

$$\mathbf{16} \;\longrightarrow\; \mathbf{16}\otimes\mathbf{1}, \qquad 16 \;=\; 16\times 1, \tag{I.9.29}$$

one Majorana spinor of $\mathrm{Spin}(1,8)$, which is exactly the minimal $9d$ spinor: 16 real components off-shell, 8 on-shell.

(f) The tally. The on-shell counting closes:

$$\underbrace{(9-2)}_{A_\mu} + \underbrace{1}_{\phi^9} \;=\; 7+1 \;=\; 8 \;=\; \underbrace{16/2}_{\lambda}, \tag{I.9.30}$$

and the Coulomb branch, once we open it in §I.9.4, will have real dimension $(10-9)\,r = r$. The result of the step, listed:

- *The $9d$ $\mathcal{N} = 1$ vector multiplet* (sixteen supercharges; the $10d$ vector on one circle). Field content: gauge field A_μ, $\mu = 0, \ldots, 8$ (7 on-shell polarizations); 1 real adjoint scalar ϕ^9; one Majorana gaugino of $\mathrm{Spin}(1,8)$ with 16 real components off-shell, 8 on-shell. R-symmetry: none (the transverse $SO(1)$ is trivial; the scalar is an $SO(1)$ singlet). Role: the one-scalar counting row between the parent and the first R-symmetric rows.

Worked instance: the $9d \to 8d$ step

Compactify x^8 on a circle of radius R_8; equivalently, run the parent on T^2. The same six displays repeat with one new phenomenon: the first commutator.

(a, b) Split and components. The $9d$ gauge field splits again,

$$A_{\hat\mu} \;\longrightarrow\; \big(A_\mu,\ \phi^8\big), \qquad \hat\mu = 0, \ldots, 8, \quad \mu = 0, \ldots, 7, \tag{I.9.31}$$

so the accumulated scalars are (ϕ^8, ϕ^9), count $10 - 8 = 2$, an $SO(2)$ vector. Relative to the $10d$ parent the 45 components of F_{MN} now sit as

$$45 \;=\; \underbrace{28}_{F_{\mu\nu}} \;+\; \underbrace{16}_{F_{\mu m}} \;+\; \underbrace{1}_{F_{89}}, \tag{I.9.32}$$

$\binom{8}{2} = 28$ gauge components, $8 \times 2 = 16$ covariant derivatives, and, for the first time, one purely transverse component,

$$F_{89} \;=\; -\,i\,[\phi^8, \phi^9]. \tag{I.9.33}$$

(c) The first potential. That single component feeds the master split (I.9.20) its first commutator-squared term, minus the potential

$$V \;=\; -\frac{1}{2g_8^2}\,\mathrm{Tr}\,[\phi^8, \phi^9]^2 \;=\; \frac{1}{2g_8^2}\,\mathrm{Tr}\Big([\phi^8, \phi^9]^\dagger [\phi^8, \phi^9]\Big) \;\geq\; 0, \tag{I.9.34}$$

non-negative because the commutator of Hermitian matrices is anti-Hermitian. At $9d$ there was no potential at all; from $8d$ down, supersymmetric vacua must make the commutators vanish, the seed of the Coulomb-branch analysis of §I.9.4.

(d) The complex scalar. The transverse rotation $SO(2)$ rotates (ϕ^8, ϕ^9) as a doublet, which packages naturally as one complex adjoint scalar of definite charge,

$$\phi \;\equiv\; \phi^8 + i\,\phi^9, \qquad \phi \;\to\; e^{i\alpha}\,\phi \quad \text{under } SO(2) = U(1)_R, \tag{I.9.35}$$

the charge-1 complex scalar of the $8d$ row (its conjugate carries charge -1).

(e) The fermion split. Under $\mathrm{Spin}(1,7) \times \mathrm{Spin}(2)$ the **16** decomposes into two Weyl blocks,

$$\mathbf{16} \;\longrightarrow\; \mathbf{8}_s^{+1/2} \,\oplus\, \mathbf{8}_c^{-1/2}, \qquad 16 \;=\; 8 + 8, \tag{I.9.36}$$

the $10d$ Weyl condition locking the $8d$ chirality to the $\mathrm{Spin}(2) = U(1)_R$ charge $\pm\frac{1}{2}$. Here α is the geometric $SO(2)$ angle of (I.9.35): it is 2π-periodic on the scalar, whereas its spin lift is 4π-periodic and acts on the fermions by $e^{\pm i\alpha/2}$. If the $\mathrm{Spin}(2)$ parameter has period 2π, the same fermions have integral charges ± 1 and the scalar has charge 2. The two blocks are exchanged by complex conjugation, so together they form one Majorana gaugino of $\mathrm{Spin}(1,7)$, the minimal $8d$ spinor: 16 real components off-shell, 8 on-shell.

(f) Coupling and tally. The second circle shifts the coupling dimension by one more unit,

$$\frac{1}{g_8^2} \;=\; \frac{2\pi R_8}{g_9^2}, \qquad [g_8^2] \;=\; -5 + 1\cdot 1 \;=\; -4 \;=\; 4 - 8, \tag{I.9.37}$$

and the on-shell tally closes again,

$$\underbrace{(8-2)}_{A_\mu} + \underbrace{2}_{\phi^8,\,\phi^9} \;=\; 6 + 2 \;=\; 8. \tag{I.9.38}$$

- *The $8d$ $\mathcal{N} = 1$ vector multiplet* (sixteen supercharges; the $10d$ vector on T^2). Field content: gauge field A_μ, $\mu = 0, \ldots, 7$ (6 on-shell polarizations); 2 real adjoint scalars (ϕ^8, ϕ^9), the vector of $SO(2)$, equivalently one complex scalar $\phi = \phi^8 + i\phi^9$ of $U(1)_R$ charge 1; one Majorana gaugino of $\mathrm{Spin}(1,7)$, decomposing as the Weyl pair $\mathbf{8}_s^{+1/2} \oplus \mathbf{8}_c^{-1/2}$ under $U(1)_R$, 16 real off-shell, 8 on-shell. R-symmetry: $U(1)_R = SO(2)$. Role: the first R-symmetric row; the field-theory home of the F-theory 7-brane gauge theory (§I.9.5).

Worked instance: the $8d \to 7d$ step

Compactify x^7 on a circle of radius R_7; cumulatively, the parent on T^3.

(a, b) Split and components.

$$A_{\hat{\mu}} \;\longrightarrow\; \big(A_\mu,\ \phi^7\big), \qquad \hat{\mu} = 0,\dots,7, \quad \mu = 0,\dots,6, \tag{I.9.39}$$

with accumulated scalars (ϕ^7, ϕ^8, ϕ^9), count $10 - 7 = 3$, a vector of $SO(3)$. The 45 components of the parent field strength now split as

$$45 \;=\; \underbrace{21}_{F_{\mu\nu}} \;+\; \underbrace{21}_{F_{\mu m}} \;+\; \underbrace{3}_{F_{mn}}, \tag{I.9.40}$$

$\binom{7}{2} = 21$, $7 \times 3 = 21$, and $\binom{3}{2} = 3$ transverse pairs, which feed the potential all three of its commutators,

$$V \;=\; \frac{1}{2g_7^2} \sum_{m<n} \mathrm{Tr}\Big([\phi^m, \phi^n]^\dagger [\phi^m, \phi^n]\Big), \qquad (m,n) \in \{(7,8),\ (8,9),\ (7,9)\}. \tag{I.9.41}$$

(c) The fermion split. Under $\mathrm{Spin}(1,6) \times \mathrm{Spin}(3)$ the **16** factorizes,

$$\mathbf{16} \;\longrightarrow\; (\mathbf{8},\, \mathbf{2}), \qquad 16 \;=\; 8 \times 2, \tag{I.9.42}$$

the Dirac spinor **8** of $\mathrm{Spin}(1,6)$ times the doublet **2** of $\mathrm{Spin}(3) = SU(2)$. Counted naively this is $8 \times 2 = 16$ *complex* components; a symplectic-Majorana condition, a reality condition that uses the doublet index, halves it back to 16 real components. The $7d$ gaugino is therefore an $SU(2)_R$ doublet λ^i, $i = 1, 2$, with 16 real components off-shell and 8 on-shell, and the doublet index is the first sighting of the $7d$ R-symmetry acting on fermions through the *cover* of $SO(3)$ (§I.9.3).

(d) Coupling and tally.

$$\frac{1}{g_7^2} \;=\; \frac{2\pi R_7}{g_8^2}, \qquad [g_7^2] \;=\; -4 + 1 \;=\; -3 \;=\; 4 - 7, \tag{I.9.43}$$

$$\underbrace{(7-2)}_{A_\mu} + \underbrace{3}_{\phi^m} \;=\; 5 + 3 \;=\; 8. \tag{I.9.44}$$

(e) The terminal Lagrangian, assembled. Collecting every piece the three steps produced gives the full $7d$ $\mathcal{N} = 1$ SYM Lagrangian, the worldvolume theory the $7d$ corresponding theory reads:

$$\begin{aligned} \mathcal{L}_7 \;=\; & -\frac{1}{4g_7^2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g_7^2}\mathrm{Tr}\, D_\mu\phi^m D^\mu\phi^m + \frac{1}{4g_7^2}\mathrm{Tr}\,[\phi^m,\phi^n][\phi^m,\phi^n] \\ & + \frac{i}{2g_7^2}\mathrm{Tr}\,\bar{\lambda}\,\Gamma^\mu D_\mu\lambda + \frac{1}{2g_7^2}\mathrm{Tr}\,\bar{\lambda}\,\Gamma^m[\phi^m,\lambda], \end{aligned} \tag{I.9.45}$$

written in $10d$ spinor notation (λ the reduced gaugino components, Γ^m the surviving internal gamma matrices); regrouping the fermions into the $SU(2)_R$ doublet is the split (I.9.42). Every coefficient is the master split (I.9.20) plus the Yukawa pattern of (I.9.28), now with all three scalars.

- *The* $7d$ $\mathcal{N}=1$ *vector multiplet* (sixteen supercharges; the $10d$ vector on T^3). Field content: gauge field A_μ, $\mu=0,\ldots,6$ (5 on-shell polarizations); 3 real adjoint scalars (ϕ^7,ϕ^8,ϕ^9), the vector of $SO(3)$, equivalently a triplet of $SU(2)_R$; one symplectic-Majorana gaugino doublet λ^i of $\mathrm{Spin}(1,6)$, the $(\mathbf{8},\mathbf{2})$ of $\mathrm{Spin}(1,6)\times SU(2)_R$, 16 real off-shell, 8 on-shell. R-symmetry: $SU(2)_R$, the spinor cover of the transverse $SO(3)$. Role: the terminal row of this chapter; the field-theory home of M-theory on ADE singularities (§I.9.5).

Composing the steps. The three circles compose into one statement: reducing directly on the torus T^{10-d} divides the parent coupling by the torus volume,

$$\frac{1}{g_d^2} \;=\; \frac{\mathrm{Vol}\,(T^{10-d})}{g_{10}^2}, \qquad [g_d^2] \;=\; -6+(10-d) \;=\; 4-d, \tag{I.9.46}$$

with $\mathrm{Vol}\,(T^{10-d}) = \prod_m 2\pi R_m$ carrying mass dimension $-(10-d)$. The one-step relations (I.9.27), (I.9.37), (I.9.43) are this formula peeled one circle at a time.

The reduction dictionary at a glance

Per step $d+1\to d$: split the gauge field (I.9.16); the new scalar's kinetic term is the mixed field strength (I.9.19); commutators of accumulated scalars feed the potential (I.9.20); the circle volume shifts $[g^2]$ up by one (I.9.46); the 16 gaugino components repackage under $\mathrm{Spin}(1,d-1)\times\mathrm{Spin}(10-d)$; and the bosonic tally $(d-2)+(10-d)=8$ closes. Everything else in the chapter reads off this dictionary.

The four rows, collected. Reading the three steps top-down gives the four rows this chapter owns, collected in Table I.9.1: $10d\,\mathcal{N}=1\to 9d\,\mathcal{N}=1\to 8d\,\mathcal{N}=1\to 7d\,\mathcal{N}=1$, with $10-d=0,1,2,3$ adjoint scalars. The descent continues below $7d$ to $6d$ $(1,1)$, $5d$ $\mathcal{N}=2$, and $4d$ $\mathcal{N}=4$, all still sixteen-supercharge reductions of the same parent, but those rows belong to their own flavor-family chapters (the $6d$ neighbor of Chapter 8, the $5d$ neighbor of Chapter 7, the $4d$ neighbor of Chapter 5, where $4d$ $\mathcal{N}=4$ is the maximal $4d$ superconformal theory, an interacting fixed point that does exist below seven dimensions). This chapter stops the descent at $7d$.

The coupling-dimension summary. Table I.9.2 collects the quantities determined by the number of reduced directions. The coupling has negative mass dimension in every row: each circle raises $[g^2]$ by exactly one unit (the circle volume in (I.9.27), (I.9.37), (I.9.43)), and starting from -6 three steps cannot reach zero,

$$[g^2] \;=\; 4-d \;=\; -6,\ -5,\ -4,\ -3 \qquad \text{for } d=10,\ 9,\ 8,\ 7, \tag{I.9.47}$$

strictly negative for all four rows. The pure SYM Lagrangian is perturbatively non-renormalizable in every row. String and M-theory provide ultraviolet settings for the particular gauge sectors described below. What grows as one descends is not the coupling behavior but the matter and moduli data: the scalar count $10-d$, the R-symmetry $SO(10-d)$, and the Coulomb-branch real dimension $(10-d)\,r$ all increase as the number of reduced directions grows. The next two sections derive the R-symmetry column and the Coulomb-branch column.

d	gauge field	scalars $10-d$	gaugino (16 real, decomposed)	bosonic dof
10	A_μ (8 polns)	0	Majorana–Weyl, $\Gamma_{11}\lambda = +\lambda$	$8+0=8$
9	A_μ (7 polns)	1	Majorana of Spin$(1,8)$: $16 = 16 \times 1$	$7+1=8$
8	A_μ (6 polns)	2	Weyl pair $\mathbf{8}_s^{+1/2} \oplus \mathbf{8}_c^{-1/2}$: $16 = 8+8$	$6+2=8$
7	A_μ (5 polns)	3	sympl. Majorana $(\mathbf{8},\mathbf{2})$: $16 = 8 \times 2$	$5+3=8$

Table I.9.1: The $10d \to 7d$ reduction sequence. Each row is the same sixteen-supercharge vector multiplet: the gauge field keeps $d-2$ on-shell polarizations, the $10-d$ lost components become adjoint scalars, the 16 gaugino components repackage under Spin$(1,d-1) \times$ Spin$(10-d)$, and the bosonic total $(d-2)+(10-d)=8$ is fixed, matched by 8 on-shell gaugino components at every row. Every entry is a function of d, not a per-row input.

Why one reduction determines all four theories. The value of this counting is that it *is* the sixteen-supercharge reduction sequence that Chapter 1's master table quotes and that the later volumes consume. Chapter 1 tabulates the R-symmetries and the supercharge counts; this section shows they are consequences of one operation applied to one parent. A reader handed a $9d$, $8d$, or $7d$ $\mathcal{N}=1$ SYM theory can now reconstruct its multiplet content from the row alone, and, more to the point, can re-derive it: every display in the three worked steps used nothing but the zero-mode rule $\partial_m = 0$ and index counting.

I.9.3 R-symmetry across dimensions $SO(10-d)$

The internal gauge-field components ϕ^m carry the vector index of the $10-d$ reduced directions. In the zero-mode Lagrangian with equal internal metric scales, rotations among these directions act on the scalars. The supercharges transform as spinors under the same rotations, which identifies the continuous R-symmetry algebra of the reduced theory. A generic compact torus does not itself have this full continuous isometry; the rotation statement concerns the local zero-mode theory or the decompactified internal geometry. At the algebra level the result is

$$\mathcal{R}(d) \;=\; SO(10-d) \tag{I.9.48}$$

at the algebra level. The representation assignments are uniform down the whole reduction sequence, and it is worth displaying the rule once, in one line:

$$\phi^m \in \text{vector of } SO(10-d), \qquad \lambda,\ Q \in \text{spinor of Spin}(10-d), \tag{I.9.49}$$

scalars see the rotation group, fermions see its spin cover, with the low-rank covers Spin$(2) = U(1)$ and Spin$(3) = SU(2)$. Reading the algebra data down the reduction

d	scalars $10-d$	$[g^2]=4-d$	R-symmetry $SO(10-d)$	Coulomb dim $(10-d)\,r$
10	0	-6	$SO(0)$ (none)	0
9	1	-5	$SO(1)$ (none)	r
8	2	-4	$SO(2)=U(1)_R$	$2r$
7	3	-3	$SO(3)$, cover $SU(2)_R$	$3r$

Table I.9.2: The per-row summary of the sixteen-supercharge SYM reduction sequence, $r=\operatorname{rank} G$. The coupling dimension $[g^2]=4-d$ is strictly negative for every row, so the non-renormalizability is uniform down the whole $\geq 7d$ reduction sequence, not $10d$-only. The scalar count, R-symmetry, and Coulomb-branch dimension all grow as $10-d$ increases. Each entry is computed from d (and, for the last column, the rank).

sequence makes the pattern concrete. The dimension and rank of $so(n)$ are

$$\dim\, so(n) \;=\; \frac{n(n-1)}{2}, \qquad \operatorname{rank} so(n) \;=\; \left\lfloor \frac{n}{2} \right\rfloor, \tag{I.9.50}$$

so with $n=10-d$ we get, for $d=10,9,8,7$,

$$so(0), \quad so(1), \quad so(2), \quad so(3), \qquad \dim = 0,\ 0,\ 1,\ 3. \tag{I.9.51}$$

The two top rows are trivial. At $d=10$ the transverse rotation group is $SO(0)$, the trivial group, which is why the parent has no continuous R-symmetry. At $d=9$ the transverse group is $SO(1)$, again trivial (a single direction has no rotations), so the $9d$ row also has no continuous R-symmetry; the one scalar of the $9d$ row is just counting, carried but not rotated. The R-symmetry turns on at $d=8$.

The $8d$ row: $U(1)_R$. At $d=8$ there are two reduced directions, so the transverse rotation group is $SO(2)$, the rotations of a plane. This is an abelian group, isomorphic to $U(1)$, with rank 1 and dimension 1:

$$\mathcal{R}(8) \;=\; SO(2) \;\cong\; U(1)_R, \qquad (\text{rank}, \dim) = (1,1). \tag{I.9.52}$$

The single generator rotates the two scalars ϕ^8, ϕ^9 into each other, equivalently rotates the complex combination $\phi=\phi^8+i\phi^9$ of (I.9.35) by a phase, and acts on the gaugino's two Weyl blocks with the charges $\pm\frac{1}{2}$ of (I.9.36): the half-integer fermion charges against the integer scalar charge are precisely the spin-cover rule (I.9.49) at work. The $8d$ $\mathcal{N}=1$ SYM theory carries $U(1)_R$, and, as the catalog of §I.9.5 records, this is the field-theory home of the F-theory 7-brane gauge theory (whose worldvolume is $7+1=8$-dimensional).

The $7d$ row: $SU(2)_R$, the spinor cover. At $d=7$ there are three reduced directions, so the transverse rotation group is $SO(3)$, with rank 1 and dimension 3. But the R-symmetry group is not literally $SO(3)$. The R-symmetry is the automorphism group

acting on the supercharges, and the supercharges are spinors; spinors transform under the double cover of the rotation group, which for $SO(3)$ is

$$SU(2) \;=\; \mathrm{Spin}(3) \;\longrightarrow\; SO(3), \qquad so(3) \;\cong\; su(2) \text{ as Lie algebras,} \tag{I.9.53}$$

the isomorphism holding at the level of the Lie algebra (both have rank 1 and dimension 3) while the groups differ by the $\mathbb{Z}_2$ center. The genuine R-symmetry of $7d$ $\mathcal{N}=1$ SYM is the spinor cover,

$$\mathcal{R}(7) \;=\; SU(2)_R, \tag{I.9.54}$$

the master-table row "$SO(3)$ transverse, double-covered by $SU(2)$." This is the chapter's central structural statement: the transverse rotation group of the reduced dimensions becomes the R-symmetry, and the cover to $SU(2)$ passes exactly at $7d$, where $SO(3)$ first appears. The three adjoint scalars ϕ^7, ϕ^8, ϕ^9 form a vector of $SO(3)$, equivalently a triplet of $SU(2)_R$; the gaugino forms the doublet λ^i that the $8d \to 7d$ step (I.9.42) already displayed. The same rule keeps working below this chapter's range, which is a useful cross-check against a row the reader already owns: continuing the reduction to $d=4$ gives six reduced directions and

$$\mathrm{Spin}(6) \;=\; SU(4), \qquad so(6) \;\cong\; su(4), \quad (\mathrm{rank}, \dim) = (3, 15), \tag{I.9.55}$$

the $SU(4)_R$ of $4d$ $\mathcal{N}=4$ SYM quoted in Chapters 1 and 5, produced by the identical transverse-rotation logic.

Common misconception: the $7d$ $\mathcal{N}=1$ R-symmetry is $SO(3)$

The transverse rotation group of the three reduced directions is $SO(3)$, and it rotates the three adjoint scalars ϕ^m as a vector. But the R-symmetry is the automorphism of the *supercharges*, and the supercharges are spinors: they transform under the double cover $SU(2) = \mathrm{Spin}(3)$, not $SO(3)$. So the $7d$ R-symmetry is $SU(2)_R$. The cover is physical, not a labeling choice, because the gaugino sits in the spinor (doublet) representation, so the genuine R-symmetry group is $SU(2)$. The same pattern is why $4d$ $\mathcal{N}=4$ has $SU(4)_R = \mathrm{Spin}(6)$ and not $SO(6)$: whenever the reduced rotation group acts on spinors, the R-symmetry is its spinor cover.

I.9.4 The Coulomb branch on reduction

The parent has no Coulomb branch because it has no adjoint scalar. Reduction produces $10-d$ adjoint scalars ϕ^m, and their vacuum expectation values open a Coulomb branch. We work out which configurations are supersymmetric vacua and count the resulting moduli.

The flat-direction condition. The scalars ϕ^m inherit their interactions from the $10d$ gauge kinetic term: the last term of the master split (I.9.20) is minus the scalar potential, which the $9d \to 8d$ step derived with its coefficient (I.9.34). In general,

$$V(\phi) \;=\; -\frac{1}{2g_d^2}\sum_{m<n} \mathrm{Tr}\,[\phi^m, \phi^n]^2 \;=\; \frac{1}{2g_d^2}\sum_{m<n} \mathrm{Tr}\Big([\phi^m, \phi^n]^\dagger [\phi^m, \phi^n]\Big) \;\geq\; 0, \tag{I.9.56}$$

a sum of squared commutator norms, non-negative because each commutator of Hermitian matrices is anti-Hermitian. The fixed Pauli-matrix exercise computes an ordinary matrix Frobenius norm; a physical coefficient in V additionally uses the declared generator trace and coupling normalization. A supersymmetric vacuum is a zero-energy configuration, $V = 0$, which because each term is a norm forces every commutator to vanish:

$$[\phi^m, \phi^n] \;=\; 0 \qquad \text{for all } m, n. \tag{I.9.57}$$

The flat directions are the *commuting* configurations of the $10 - d$ adjoint scalars: sets of matrices that can be simultaneously diagonalized. Simultaneously diagonalizable adjoint matrices lie, up to a gauge rotation, in a common Cartan subalgebra. So the vacuum moduli are the $10 - d$ scalars each taking a value in the Cartan of G, modulo the residual gauge identifications, which for the Cartan is the Weyl group W_G. For a local reduction, these coordinates give the real Coulomb space

$$\mathcal{M}_{\text{Coulomb}}^{\text{local}} \;=\; \mathbb{R}^{(10-d)\,r}\big/W_G, \qquad r = \operatorname{rank} G, \tag{I.9.58}$$

each of the $10 - d$ scalars contributing its r Cartan directions. On a genuinely compact torus, a large gauge transformation can shift a Cartan component by a cocharacter divided by its circle radius. The global Wilson-line space therefore has further periodic identifications. The displayed real quotient is the local scalar description, which is the one used for the dimension count and the brane interpretations below.

The dimension, and the generic breaking. The real dimension of the Coulomb branch is therefore

$$\dim_{\mathbb{R}} \mathcal{M}_{\text{Coulomb}} \;=\; (10 - d)\, r, \tag{I.9.59}$$

the number of reduced directions times the rank. At a generic point the adjoint scalars have distinct Cartan eigenvalues and Higgs the gauge group to its maximal torus,

$$G \;\longrightarrow\; U(1)^r, \qquad \#\,\text{massive gauge bosons} \;=\; \dim G - r \;\big(= N^2 - N \text{ for } SU(N)\big). \tag{I.9.60}$$

The mechanism is visible in the commutators. A Cartan-valued vev acts diagonally on the root generators E_α,

$$[\phi^m, E_\alpha] \;=\; \alpha(a_m)\, E_\alpha \qquad \text{for } \phi^m \text{ in the Cartan with eigenvalue data } a_m, \tag{I.9.61}$$

so the off-Cartan modes are charged under the surviving $U(1)^r$ and pick up masses proportional to the vevs through the commutator terms of (I.9.20), while the r Cartan modes commute with everything and stay massless: exactly the $\dim G - r$ count of (I.9.60). At the parent $d = 10$ the formula (I.9.59) gives $(10 - 10)\, r = 0$: no Coulomb branch, consistent with the absence of any adjoint scalar. The dimension grows as one descends, since $10 - d$ grows: r at $9d$, $2r$ at $8d$, $3r$ at $7d$. This is the last column of Table I.9.2.

The branch is real. The Coulomb branch here is a *real* moduli space, a space of commuting Hermitian matrices modulo Weyl. This is the high-dimensional analog of the $5d$ $\mathcal{N} = 1$ real Coulomb branch of Chapter 7, where the single real vector-multiplet scalar likewise takes real Cartan values. It is *not* the complex special-Kähler Coulomb

branch of $4d$ $\mathcal{N}=2$ (Chapter 5), whose scalar is a complex combination protected into a rigid special geometry, and it is *not* a hyperkähler Higgs branch: a Higgs branch needs hypermultiplets, and pure super Yang–Mills has none. The reduced scalars are real adjoint fields, and their moduli space is flat and real, the simplest kind of Coulomb branch.

Worked instance: the $7d$ $\mathcal{N}=1$ $SU(N)$ Coulomb branch

Take $7d$ $\mathcal{N}=1$ SYM with gauge group $SU(N)$. The reduced-direction count is $10-7=3$, so there are three adjoint scalars, the $SO(3)$ vector, equivalently the $SU(2)_R$ triplet. The rank of $SU(N)$ is $r=N-1$. The Coulomb-branch real dimension is

$$\dim_{\mathbb{R}} \mathcal{M}_{\text{Coulomb}} \;=\; (10-7)(N-1) \;=\; 3(N-1), \tag{I.9.62}$$

three real Cartan directions per unbroken $U(1)$. For $SU(2)$, $r=1$ and the branch is $3(2-1)=3$-dimensional: three real scalars, the $SU(2)_R$ triplet of the single Cartan direction. For $SU(3)$, $r=2$ and the branch is $3(3-1)=6$-dimensional. A generic point Higgses $SU(N)$ to its maximal torus $U(1)^{N-1}$, giving mass to N^2-N gauge bosons (2 for $SU(2)$, 6 for $SU(3)$, per (I.9.60)).

The flat-direction condition is concrete at $SU(2)$. Write the three adjoint scalars as 2×2 traceless Hermitian matrices, and align them all with the Cartan generator σ_3:

$$\phi^m \;=\; a_m\,\sigma_3, \qquad m=7,8,9, \quad a_m\in\mathbb{R}. \tag{I.9.63}$$

Every commutator $[\phi^m,\phi^n]=a_m a_n[\sigma_3,\sigma_3]=0$ vanishes, the potential $V=0$, and the configuration is a supersymmetric vacuum: three real moduli a_7,a_8,a_9, matching $3(N-1)=3$. The root action (I.9.61) is explicit here: with $\sigma_\pm=(\sigma_1\pm i\sigma_2)/2$,

$$[\phi^m,\sigma_\pm] \;=\; \pm 2a_m\,\sigma_\pm, \tag{I.9.64}$$

the two off-Cartan generators are charged and become the two massive gauge bosons of (I.9.60). If instead two of the scalars point along non-commuting generators, say $\phi^7=\sigma_1$ and $\phi^8=\sigma_2$, then

$$[\sigma_1,\sigma_2] \;=\; 2i\,\sigma_3 \;\neq\; 0, \tag{I.9.65}$$

the potential is strictly positive, and the configuration is *not* a vacuum. Only the commuting, Cartan-aligned configurations survive, exactly the $\mathbb{R}^{3(N-1)}/W$ count.

> **Common misconception: high-dimensional SYM has a rank-dimensional Coulomb branch**
>
> $10d$ $\mathcal{N}=1$ SYM has *no* Coulomb branch at all: the gauge field has no adjoint-scalar superpartner. The Coulomb branch appears only on dimensional reduction, where the $10-d$ gauge-field components along the reduced directions become $10-d$ adjoint scalars. Its real dimension is therefore $(10-d)\,r$, the number of reduced directions times the rank, which *grows* as you descend, not a fixed r. And it is a *real* moduli space of commuting Hermitian matrices, the high-dimensional analog of the $5d$ real Coulomb branch, not the complex special-Kähler branch of $4d$ $\mathcal{N}=2$ and not a hyperkähler Higgs branch (pure SYM has no hypermultiplets). The flat directions are the simultaneously diagonalizable configurations $[\phi^m,\phi^n]=0$, Higgsing $G\to U(1)^r$ at a generic point.

I.9.5 The per-dimension catalog and string or M-theory setting

The four rows can now be read off with their R-symmetries and their string or M-theory homes. Table I.9.3 collects them. The R-symmetry entry in each row is the grounding computation of §I.9.3, $SO(10-d)$ with the $7d$ cover; the final column is a later-use pointer, naming the construction that reads its field content off this row, not something built here.

d	R-symmetry	content	string or M-theory setting (later-use pointer)
10	none ($SO(0)$)	A_μ, 0 scalars	heterotic / type I gauge sector
9	none ($SO(1)$)	A_μ, 1 scalar	the one-scalar counting row; no separate construction discussed here
8	$U(1)_R$ ($SO(2)$)	A_μ, 2 scalars	F-theory 7-brane gauge theory (world-volume $8d$)
7	$SU(2)_R$ ($SO(3)$ cover)	A_μ, 3 scalars	M-theory on ADE / G_2-holonomy singularities

Table I.9.3: The per-dimension catalog of the $7d$-$10d$ sixteen-supercharge SYM rows. The R-symmetry is $SO(10-d)$ (trivial at $9d/10d$, $U(1)_R$ at $8d$, the $SU(2)_R$ cover at $7d$); the content is the $10d$ vector reduced, A_μ plus $10-d$ adjoint scalars; the final column identifies settings in which these field theories appear; their construction lies beyond this chapter.

The $10d$ row supplies the classical gauge-field content of the heterotic string and of type I. Its vector multiplet has $\Gamma_{11}\lambda = +\lambda$; the anomalous standalone nonabelian action is not the complete quantum gauge sector. In the string backgrounds the gauge fields couple to additional string and gravity fields, with anomaly cancellation imposed on the full system. The next two examples explain where the reduced multiplets occur in string and M-theory.

Why the $8d$ row matters: F-theory 7-branes. A 7-brane has a $7+1=8$-dimensional worldvolume, so its gauge theory lives in the $8d$ row (not the $7d$ row: the "7" counts spatial worldvolume dimensions, an off-by-one this row-labeling prevents). The brane sits at a point in the two transverse directions, and the row's field content is the geometry of that statement:

$$\phi \;=\; \phi^8 + i\,\phi^9 \;\;\longleftrightarrow\;\; z_\perp, \qquad U(1)_R \;=\; SO(2)_\perp, \tag{I.9.66}$$

the complex adjoint scalar is the brane's complex transverse position, and the $U(1)_R$ we derived is the rotation of the transverse plane. In F-theory the 7-brane's gauge group is read off from the singularity type of an elliptic fibration; what the construction consumes from this chapter is exactly the $8d$ row of Table I.9.3: multiplet, $U(1)_R$, and the $2r$-dimensional Coulomb branch of transverse positions.

Why the $7d$ row matters: M-theory on ADE singularities. A $7d$ $\mathcal{N}=1$ gauge theory arises locally on the singular locus of an ADE ALE fiber in M-theory: eleven dimensions minus the four real dimensions of the fiber leave a $7d$ worldvolume. For each of the $r=\mathrm{rank}(G)$ independent exceptional two-cycles C_a, the three Cartan-scalar directions encode its hyperkähler resolution parameters,

$$\vec{\phi}_a \;\longleftrightarrow\; \left(\int_{C_a}\omega_1,\ \int_{C_a}\omega_2,\ \int_{C_a}\omega_3\right), \qquad a=1,\dots,r, \qquad SU(2)_R \;=\; \mathrm{Spin}(3)_\omega, \tag{I.9.67}$$

where ω_i are the three hyperkähler two-forms. The proportionality between these geometric periods and canonically normalized fields depends on the M-theory reduction and is not computed here. Their $SO(3)$ rotation has the displayed spin cover on supercharges. At generic local resolution the Cartan theory has $U(1)^r$; M2-branes wrapped on exceptional cycles become massless as the cycles collapse, restoring the ADE gauge symmetry. The G_2-holonomy compactifications that engineer $4d$ physics pass through this local $7d$ description, but global resolution and dynamics require additional input.

I.9.6 No superconformal field theory above six dimensions

The final structural fact removes this whole dimension range from the extremization through-line that organized Chapters 3, 4, and 6. There is no interacting superconformal field theory in $d \geq 7$ at sixteen supercharges. Consequently there is nothing here to extremize: no a-maximization, no c-extremization, no F-maximization, and no conformal central charges a, c to compute, because there is no conformal fixed point to compute them at.

The superconformal-algebra classification recalled in Chapter 1 excludes an interacting fixed point with these supercharges above six dimensions. This algebraic result sets the scope. The coupling calculation in §I.9.2 separately shows why the pure Yang–Mills Lagrangian requires ultraviolet input:

$$[g^2] \;=\; 4-d \;\leq\; -3 \;<\; 0 \qquad \text{for every } d \geq 7: \tag{I.9.68}$$

the perturbative Yang–Mills description is non-renormalizable. Negative coupling dimension alone does not exclude a nonperturbative ultraviolet fixed point. The absence of a superconformal algebra is the separate reason no interacting superconformal completion occurs in these dimensions. The string and M-theory settings named above provide ultraviolet settings for particular gauge sectors; their dynamics are developed elsewhere.

This is why the chapter has no extremization member and closes the dimensional reduction sequence. The rich extremization structure of the lower dimensions, the a-maximization anchor of Chapter 3 and its $2d$ and $3d$ analogs, has no counterpart here because it presupposes a fixed point that does not exist at these dimensions. Specific lower-dimensional examples include $4d$ $\mathcal{N}=4$ SYM and the $6d$ $(2,0)$ theory, which have

superconformal fixed points. Five-dimensional maximal SYM instead has a dimensionful coupling and a six-dimensional $(2,0)$ completion; it does not give a five-dimensional interacting fixed point. Chapter 1's $32Q$ ceiling caps the reduction sequence above $10d$ with supergravity, and this chapter's $11d$ counting (I.9.7) is the same ceiling seen in degrees of freedom; Nahm's $d = 6$ ceiling caps the superconformal theories below; and this chapter fills the $7d$-$10d$ gap between them with the non-renormalizable pure-SYM interface, which is exactly the field theory the string and M-theory constructions start from.

Exit checklist

After this chapter the reader can

1. identify $10d$ $\mathcal{N} = 1$ super Yang–Mills as the maximal-dimension parent, list its vector multiplet (A_M, Majorana–Weyl gaugino, no scalars), run the counting chain $64 \to 32 \to 16 \to 8$ against the $D - 2 = 8$ polarizations to verify $8 = 8$, and compute the coupling dimension $[g^2] = 4 - 10 = -6 < 0$ (non-renormalizable);

2. explain why there is no $11d$ SYM *without* misusing the even-dimensional Majorana–Weyl formula: the $11d$ minimal spinor is Majorana with 32 real components, giving 16 on-shell fermionic states against 9 gauge polarizations and no available scalars, so the minimal $11d$ multiplet is the supergravity multiplet, $44 + 84 = 128 = 128$;

3. run each reduction step $d + 1 \to d$ explicitly: split the gauge field with its index ranges, track the 45 field-strength components $(36+9+0,\ 28+16+1,\ 21+21+3)$, derive $F_{\mu m} = D_\mu \phi^m$ and $F_{mn} = -i[\phi^m, \phi^n]$ from the zero-mode rule, split the action into gauge, scalar-kinetic, and commutator-potential terms, and shift the coupling dimension by one unit per circle $(-6, -5, -4, -3)$;

4. decompose the 16 gaugino components at each row, $16 = 16 \times 1$ ($9d$ Majorana), $16 = 8{+}8$ (the $8d$ Weyl pair with $U(1)_R$ charges $\pm\frac{1}{2}$), $16 = 8 \times 2$ (the $7d$ symplectic-Majorana $SU(2)_R$ doublet), and verify the preserved totals $(d - 2) + (10 - d) = 8 = 16/2$;

5. state each reduced vector multiplet as a list (gauge field, $10 - d$ real scalars, fermion content, R-symmetry representations: scalars in the vector of $SO(10 - d)$, fermions under $\mathrm{Spin}(10 - d)$, with $\mathrm{Spin}(2) = U(1)$ and $\mathrm{Spin}(3) = SU(2)$);

6. count the Coulomb branch that opens on reduction from the commuting-vev flat directions $[\phi^m, \phi^n] = 0$, giving the local real moduli space $\mathbb{R}^{(10-d)\,r}/W_G$ of dimension $(10 - d)\,r$ ($3(N - 1)$ for $7d$ $SU(N)$, 0 at $d = 10$), with $\dim G - r$ gauge bosons massive at a generic point;

7. place each row in its catalog with its string or M-theory setting (heterotic / type I at $10d$, the F-theory 7-brane theory at $8d$ with the complex transverse position $\phi = \phi^8 + i\phi^9$, M-theory on ADE / G_2 singularities at $7d$ with the scalar triplets as local ALE resolution periods), knowing these are later-use pointers routed to the construction volumes;

8. state that there is no interacting superconformal field theory at $d \geq 7$ with sixteen supercharges, from the Nahm classification; $[g^2] \leq -3 < 0$ separately shows perturbative non-renormalizability, so no extremization runs here and this chapter closes the dimensional reduction sequence.

Status and sources

The invariant trace and gauge coupling are normalized as in the displayed action.

The reduction uses the spinor and supercharge conventions of Chapter 1, the Yang–Mills dimension $[g^2] = 4 - d$, and the real Coulomb branch for the indicated gauge group. Pure nonabelian $10d$ SYM is used here as a classical reduction action; its standalone quantum gauge anomaly is not removed by this calculation (Seiberg (1998, §2)).

§I.9.1 The $10d$ parent. The itemized $10d$ $\mathcal{N} = 1$ vector multiplet (A_M, Majorana–Weyl gaugino), the Lagrangian (I.9.1), the counting chain $64 \to 32 \to 16$ (I.9.3) and the on-shell dof match $D - 2 = 8$ (I.9.2) against $16/2 = 8$ (I.9.4) giving $8 = 8$ (I.9.5), the coupling dimension $[g^2] = 4 - 10 = -6 < 0$ (I.9.9)–(I.9.11) with the cutoff scale $(g^2_{10})^{-1/6}$ (I.9.12) (non-renormalizable), the corrected $11d$ argument (the $11d$ minimal spinor is Majorana, 32 real $\to$ 16 on-shell (I.9.6), against 9 polarizations (I.9.7); the minimal $11d$ multiplet is the supergravity multiplet $44 + 84 = 128 = 128$ (I.9.8)), and the absence of a Coulomb branch or continuous R-symmetry at $d = 10$. (Brink et al. (1977) the classical $10d$ action and its reduction; Gliozzi et al. (1977) the dof match; Cremmer, Julia, and Scherk (1978) the $11d$ supergravity multiplet). Additional calculations include the dof counts computed from D (polarizations $D - 2$, the Majorana / Weyl / on-shell halving chain $64 \to 32 \to 16 \to 8$), $[g^2] = -6 < 0$, the $11d$ counting (32 real Majorana, 16 on-shell, $9 = D - 2$, $16 \neq 9$, and the supergravity match $44 + 84 = 128 = 128$, every entry computed from $D = 11$), no continuous R ($SO(0)$), no Coulomb scalar at $d = 10$.

§I.9.2 The reduction sequence. The zero-mode reduction on T^{10-d} (I.9.13)–(I.9.15), the gauge-field split $A_M \to (A_\mu, \phi^m)$ (I.9.16) with the surviving gauge transformations (I.9.17) and the scalar count $10 - d$ (I.9.18), the general component formulas $F_{\mu m} = D_\mu \phi^m$, $F_{mn} = -i[\phi^m, \phi^n]$ (I.9.19), the master action split (I.9.20), the preserved bosonic dof $(d - 2) + (10 - d) = 8$ (I.9.21); the three worked steps: $10d \to 9d$ in full ((I.9.22)–(I.9.30): the $45 = 36 + 9 + 0$ component split, $F_{\mu 9} = D_\mu \phi^9$, the action split, the coupling relation $1/g_9^2 = 2\pi R_9 / g_{10}^2$ with $[g_9^2] = -5$, the Yukawa coupling, the fermion restriction $16 = 16 \times 1$), $9d \to 8d$ ((I.9.31)–(I.9.38): $45 = 28 + 16 + 1$, the first commutator F_{89} and potential, the complex scalar, the Weyl-pair split $16 = 8 + 8$, $[g_8^2] = -4$), and $8d \to 7d$ ((I.9.39)–(I.9.44): $45 = 21 + 21 + 3$, the three commutator pairs, the symplectic-Majorana split $16 = 8 \times 2$, $[g_7^2] = -3$, the assembled terminal Lagrangian (I.9.45)); the composed torus-volume coupling relation (I.9.46); the itemized $9d/8d/7d$ vector multiplets; the field-content table (Table I.9.1) and the coupling-dimension summary $[g^2] = 4 - d = -6, -5, -4, -3$ (I.9.47) (Table I.9.2, the non-renormalizability uniform through the reductions). (Brink et al. (1977) the dimensional reduction; Seiberg (1998) the sixteen-supercharge rows). Additional calculations include the scalar count $10 - d$ and the coupling dimension $4 - d$ computed from d for every row, the per-circle shift $[g_d^2] = [g_{d+1}^2] + 1$, the 16 supercharges preserved (from the supercharge count in Chapter 1), the $(d - 2) + (10 - d) = 8$ identity at each row, the field-strength component splits $45 = 36 + 9 + 0 = 28 + 16 + 1 = 21 + 21 + 3$ (and the identity $\binom{d}{2} + d(10 - d) + \binom{10-d}{2} = 45$ symbolically in d), the commutator-pair count

$\binom{10-d}{2} = 0, 1, 3$, the fermion splits $16 = 16 \times 1 = 8 + 8 = 8 \times 2$ with the block dims computed from $2^{\lfloor d/2 \rfloor}$ and $2^{\lfloor (10-d)/2 \rfloor}$, and the reduced-Lagrangian mass-dimension homogeneity per row.

§I.9.3 R-symmetry across dimensions. The R-symmetry as the transverse rotation group $SO(10-d)$ (I.9.48), the uniform representation rule (scalars in the vector of $SO(10-d)$, fermions under $\mathrm{Spin}(10-d)$, with $\mathrm{Spin}(2) = U(1)$ and $\mathrm{Spin}(3) = SU(2)$) (I.9.49), the $so(n)$ rank/dimension formulas (I.9.50) giving the sequence $so(0), so(1), so(2), so(3)$ of dimension $0, 0, 1, 3$ (I.9.51), the $8d$ $SO(2)$ algebra and its Spin(2) cover (I.9.52) with the $\pm\frac{1}{2}$ geometric-angle fermion charges, the $7d$ $SO(3) \to SU(2)_R$ spinor cover (I.9.53)–(I.9.54), and the below this range cross-check $\mathrm{Spin}(6) = SU(4)$ at $(\mathrm{rank}, \dim) = (3, 15)$ (I.9.55). (Seiberg (1998) theories with sixteen supercharges in various dimensions). The rank and dimension follow from the stated $SO(10-d)$ formula. the $7d$ $so(3) \cong su(2)$ coincidence at $(\mathrm{rank}, \dim) = (1, 3)$ with the R-group the spinor cover $SU(2)$; the $8d$ $U(1)_R$; the $so(6) = su(4)$ coincidence at $(3, 15)$; the transverse spinor dims $2^{\lfloor (10-d)/2 \rfloor}$ behind (I.9.49).

§I.9.4 The Coulomb branch on reduction. The scalar potential with its derived coefficient $V = -\frac{1}{2g^2} \sum_{m<n} \mathrm{Tr}[\phi^m, \phi^n]^2 \geq 0$ (I.9.56) (inherited from the master split (I.9.20); the sign from the anti-Hermiticity of the commutator), the flat-direction / commuting-vev condition $[\phi^m, \phi^n] = 0$ (I.9.57), the real moduli space $\mathbb{R}^{(10-d)r}/W_G$ (I.9.58) of dimension $(10-d)\,r$ (I.9.59), the generic Higgsing $G \to U(1)^r$ with $\dim G - r$ massive gauge bosons (I.9.60) through the root action (I.9.61), the real-branch type (vs the complex / hyperkähler branches), and the worked $7d$ $SU(N)$ count $3(N-1)$ (I.9.62) ($SU(2)$: 3; $SU(3)$: 6) with the explicit Cartan configuration (I.9.63), the root eigenvalues $[\phi^m, \sigma_\pm] = \pm 2 a_m \sigma_\pm$ (I.9.64), and the non-commuting rejection $[\sigma_1, \sigma_2] = 2i\sigma_3$ (I.9.65). (Seiberg (1998) the sixteen-supercharge Coulomb branch). The worked rows give dimensions 1, 2 and 3 for $SU(2)$ in $9d$, $8d$ and $7d$, and dimension 6 for $7d$ $SU(3)$. The massive-boson count for $SU(N)$ is $N^2 - N$. The trace identity $\mathrm{Tr}(CC) = -\mathrm{Tr}(C^\dagger C)$ gives zero potential on the commuting Cartan configuration and a positive value for the noncommuting pair. The off-diagonal eigenvectors satisfy $[a\sigma_3, \sigma_\pm] = \pm 2 a \sigma_\pm$, while the Cartan mode commutes.

§I.9.5 The per-dimension catalog. The per-row catalog (Table I.9.3) of R-symmetry, content, and engine home: $10d$ heterotic / type I gauge sector; $9d$ the one-scalar row; $8d$ $U(1)_R$ the F-theory 7-brane gauge theory (worldvolume $8d$; the complex scalar as the transverse position (I.9.66)); $7d$ $SU(2)_R$ M-theory on ADE / G_2-holonomy singularities (the three scalar triplets as ALE resolution periods (I.9.67)). Sources: Vafa (1996) for the F-theory 7-brane setting; Acharya and Gukov (2004, §5.1, pp. 40–43) the local ADE/ALE cycle, three hyperkähler parameters, wrapped M2 enhancement, and $SO(3)$ action; Acharya and Witten (2001) the G_2 singular-locus setting. The local r-cycle dictionary is the claim. The period-to-canonically-normalized-scalar factor and global G_2 resolution are outside this local geometric result.

§I.9.6 No $\geq 7d$ superconformal field theory. The statement that no interacting superconformal field theory exists in $d \geq 7$ at sixteen supercharges is imported from the Nahm classification recalled from Chapter 1. The dimension estimate $[g^2] = 4 - d \leq -3$ in (I.9.68) separately establishes perturbative non-renormalizability, not a theorem excluding nonperturbative fixed points. Nahm (1978) supports the superconformal-algebra classification; Seiberg (1998, §7) distinguishes the dimensionful $5d$ maximal-SYM coupling from its $6d$ $(2,0)$ completion. This is a statement-level recall: the absence follows from the Nahm classification cited in Chapter 1. The coupling dimension is a separate calculation.

Routed (stated, not proved). Ultraviolet settings for these gauge sectors, without a universal completion theorem for arbitrary pure SYM: the anomaly-cancelled heterotic / type I string-plus-gravity sectors at $10d$, F-theory 7-branes at $8d$ (worldvolume $7 + 1 = 8d$), and

M-theory on ADE / G_2-holonomy singularities at $7d$. The lower sixteen-supercharge rows ($6d$ $(1,1)$, $5d$ $\mathcal{N}=2$, $4d$ $\mathcal{N}=4$) are pointed at their flavor-family chapters (Chapters 8, 7, 5) and not developed here. All are stated and named here; their constructions live in the engine volumes and Part III.

Further reading

The local ADE resolution-period and wrapped-M2 dictionary is in Acharya and Gukov (2004, §5.1, pp. 40–43). Ten-dimensional super Yang–Mills and its dimensional reduction are in Brink et al. (1977) and Scherk and Schwarz (1979); eleven-dimensional supergravity in Cremmer, Julia, and Scherk (1978) and Cremmer and Julia (1979), with the membrane perspective of Townsend (1995) and the Kaluza–Klein review Duff et al. (1986). The role of these maximal theories as limits of string and M-theory dynamics is discussed in Witten (1995), and the M-theory realization of gauge dynamics on G_2 and ADE singularities in Acharya (1999) and Atiyah and Witten (2003).

For modern viewpoints beyond flat-space component reduction, the twists of SYM in dimensions $2 \le d \le 10$ are classified in Elliott et al. (2022), curved-space $7d$ localization is developed in Iakovidis et al. (2020), and the $7d$ gauge-theory description of local G_2 backgrounds is studied in Barbosa et al. (2020). The pure-spinor organization of ten-dimensional on-shell data and its field-theory limit are reviewed in Mafra and Schlotterer (2023).

References

Acharya, B. S. and S. Gukov (2004). "M theory and singularities of exceptional holonomy manifolds". In: *Phys. Rept.* 392, pp. 121–189. DOI: 10.1016/j.physrep.2003.10.017. arXiv: hep-th/0409191.

Acharya, B. S. and E. Witten (2001). "Chiral fermions from manifolds of G(2) holonomy". In: arXiv: hep-th/0109152.

Acharya, B. S. (1999). "M theory, Joyce orbifolds and superYang-Mills". In: *Adv. Theor. Math. Phys.* 3, pp. 227–248. DOI: 10.4310/ATMP.1999.v3.n2.a3. arXiv: hep-th/9812205.

Atiyah, M. and E. Witten (2003). "M theory dynamics on a manifold of G(2) holonomy". In: *Adv. Theor. Math. Phys.* 6, pp. 1–106. DOI: 10.4310/ATMP.2002.v6.n1.a1. arXiv: hep-th/0107177.

Barbosa, R. et al. (2020). "T-branes and G_2 backgrounds". In: *Phys. Rev. D* 101.2, p. 026015. DOI: 10.1103/PhysRevD.101.026015. arXiv: 1906.02212 [hep-th].

Brink, L., J. H. Schwarz, and J. Scherk (1977). "Supersymmetric Yang-Mills Theories". In: *Nucl. Phys. B* 121, pp. 77–92. DOI: 10.1016/0550-3213(77)90328-5.

Cremmer, E. and B. Julia (1979). "The SO(8) Supergravity". In: *Nucl. Phys. B* 159, pp. 141–212. DOI: 10.1016/0550-3213(79)90331-6.

Cremmer, E., B. Julia, and J. Scherk (1978). "Supergravity Theory in Eleven-Dimensions". In: *Phys. Lett. B* 76, pp. 409–412. DOI: 10.1016/0370-2693(78)90894-8.

Duff, M. J., B. E. W. Nilsson, and C. N. Pope (1986). "Kaluza-Klein Supergravity". In: *Phys. Rept.* 130, pp. 1–142. DOI: 10.1016/0370-1573(86)90163-8.

Elliott, C., P. Safronov, and B. R. Williams (2022). "A taxonomy of twists of supersymmetric Yang–Mills theory". In: *Selecta Math.* 28.4, p. 73. DOI: 10.1007/s00029-022-00786-y. arXiv: 2002.10517 [math-ph].

Gliozzi, F., J. Scherk, and D. I. Olive (1977). "Supersymmetry, Supergravity Theories and the Dual Spinor Model". In: *Nucl. Phys. B* 122, pp. 253–290. DOI: 10.1016/0550-3213(77)90206-1.

Iakovidis, N. et al. (2020). "7D supersymmetric Yang–Mills on hypertoric 3-Sasakian manifolds". In: *JHEP* 06, p. 026. DOI: 10.1007/JHEP06(2020)026. arXiv: 2003.12461 [hep-th].

Mafra, C. R. and O. Schlotterer (2023). "Tree-level amplitudes from the pure spinor superstring". In: *Phys. Rept.* 1020, pp. 1–162. DOI: 10.1016/j.physrep.2023.04.001. arXiv: 2210.14241 [hep-th].

Nahm, W. (1978). "Supersymmetries and their Representations". In: *Nucl. Phys. B* 135, pp. 149–166. DOI: 10.1016/0550-3213(78)90218-3.

Scherk, J. and J. H. Schwarz (1979). "How to Get Masses from Extra Dimensions". In: *Nucl. Phys. B* 153. Ed. by A. Salam and E. Sezgin, pp. 61–88. DOI: 10.1016/0550-3213(79)90592-3.

Seiberg, N. (1998). "Notes on theories with 16 supercharges". In: *Nucl. Phys. B Proc. Suppl.* 67, pp. 158–171. DOI: 10.1016/S0920-5632(98)00128-5. arXiv: hep-th/9705117.

Townsend, P. K. (1995). "The eleven-dimensional supermembrane revisited". In: *Phys. Lett. B* 350, pp. 184–187. DOI: 10.1016/0370-2693(95)00397-4. arXiv: hep-th/9501068.

Vafa, C. (1996). "Evidence for F theory". In: *Nucl. Phys. B* 469, pp. 403–418. DOI: 10.1016/0550-3213(96)00172-1. arXiv: hep-th/9602022.

Witten, E. (1995). "String theory dynamics in various dimensions". In: *Nucl. Phys. B* 443, pp. 85–126. DOI: 10.1016/0550-3213(95)00158-O. arXiv: hep-th/9503124.